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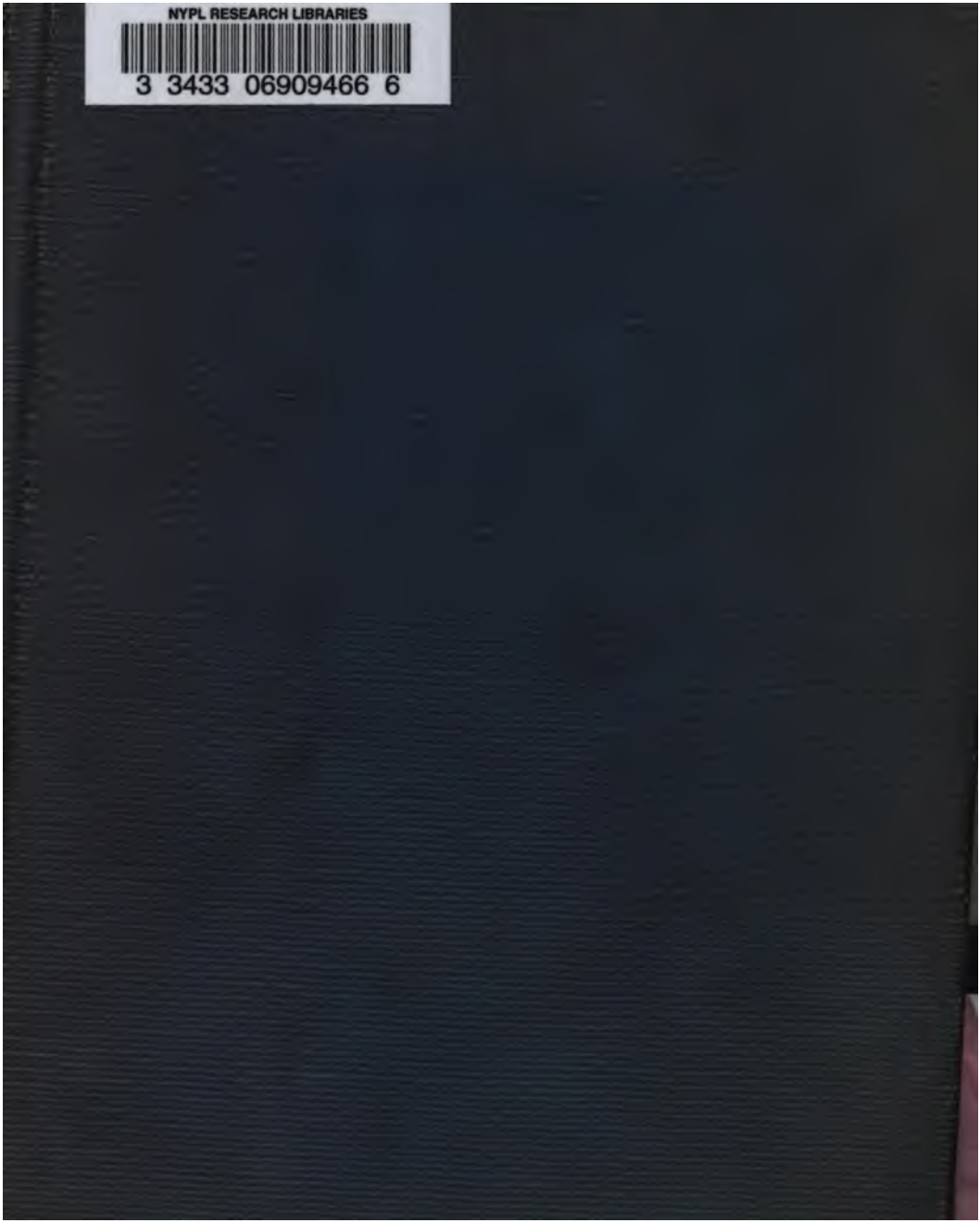
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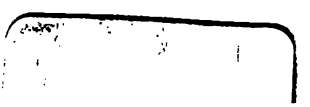

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UNIPLANAR ALGEBRA

BEING

PART I OF A PROPÆDEUTIC TO THE HIGHER MATHEMATICAL ANALYSIS

BY

IRVING STRINGHAM, Ph. D.

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PREFACE.

FROM the beginning, with rare exceptions,* a singular logical incompleteness has characterized our text-books in elementary algebra. By tradition algebra early became a mere technical device for turning out practical results, by careless reasoning inaccuracies crept into the explanation of its principles and, through compilers, are still perpetuated as current literature. Thus, instead of becoming a classic, like the geometry handed down to us from the Greeks, in the form of Euclid's *Elements*, algebra has become a collection of processes practically exemplified and of principles inadequately explained.

The labors of the mathematicians of the nineteenth century — Argand, Gauss, Cauchy, Grassmann, Peirce, Cayley, Sylvester, Kronecker, Weierstrass, G. Cantor, Dedekind and others** — have rendered unjustifiable the longer continuance of this unsatisfactory state of algebraic science. We now know what an algebra is, and the problem of its systematic unfolding into organic form is a definite and achievable one. The short treatise here presented, as the first part of a Propædeutic to the Higher Analysis, endeavors to place concisely in connected sequence the argument required for its solution.

---

* Notably, in English, Chrystal's *Algebra*, 2 vols., Edinburgh, 1886, 1889. On the continent of Europe the deficiency has been compensated mainly in works on the Higher Analysis.

** The literature through which algebra has been rehabilitated during the present century is extensive. See Stolz: *Allgemeine Arithmetik*, Leipzig, 1885, for many valuable references.

The first three chapters were made public, substantially in their present form, in a course of University Extension Lectures in San Francisco during the winter of 1891-92, a synopsis of which was issued from the University Press in October, 1891. At the close of these lectures the manuscript of the complete work was prepared for the press; but unavoidable obstacles prevented its immediate publication and a consequent delay of somewhat more than a year has intervened. This delay, however, has made possible a revision of the original sketch and some additions to its subject-matter.

The logical grounding of algebra may be attained by either of two methods, the one essentially arithmetical, the other geometrical. I have chosen the geometrical form of presentation and development, partly because of its simpler elegance, partly because this way lies the shortest path for the student who knows only the elements of geometry and algebra as taught in our schools and requires mathematical study only for its disciplinary value. The choice of method, therefore, is not to be interpreted to mean that the writer underestimates the value and the importance to the special mathematical student of the Number-System.* This system, however, has no appropriate place in the plan here presented.

The point of departure is Euclid's doctrine of proportion, and the point of view is the one that Euclid himself, could he have anticipated the modern results of mathematical science, would naturally have taken. It is interesting to note that of logical necessity the development falls mainly into the historical order. For convenience of reference the fundamental propositions of proportion are enunciated and proved in an Introduction, in which I have followed the

* Fine: *The Number-System of Algebra*, Boston, 1891.

method recommended by the Association for the Improvement of Geometrical Teaching, and published in its *Syllabus of Plane Geometry*. Except a few additions and omissions, the enunciations and numbering in Sections B and C of this Introduction are those of Hall and Stevens' admirable *Text-Book of Euclid's Elements*, Book V; and in Section D those of the *Syllabus of Plane Geometry*, Book IV, Section 2. The proofs vary in unessential particulars from those of the two texts named.

The subject-matter and treatment are such as to constitute, for the student already familiar with the elements of algebra and trigonometry, a rapid review of the underlying principles of those subjects, including in its most general aspects the algebra of complex quantities. All the fundamental formulæ of the circular and hyperbolic functions are concisely given. The chapter on Cyclometry furnishes, presumptively, a useful generalization of the circular and hyperbolic functions.

The generalized definition of a logarithm (Art. 68) and the classification of logarithmic systems,* first made public, outside of the mathematical lecture-room, in a paper read before the New York Mathematical Society in October, 1891, and subsequently published in the *American Journal of Mathematics*, are here reproduced in the revised form suggested by Professor Haskell.* A chapter on Graphical Transformations, giving the orthomorphosis of the exponential and cyclic functions, appropriately concludes this part of the subject.

Many incidental problems are suggested in the form of Agenda, useful to the student for exemplification and practice. But on the other hand, many elementary

* *American Journal of Mathematics*, Vol. XIV, pp. 187–194, and *Bulletin of the New York Mathematical Society* Vol. II, pp. 164–170.

algebraic topics are not discussed, because they are not useful to the main object of the work, and it was especially desirable that its purpose should not be hindered by the making of a large book.

A few innovations in notation and nomenclature have been unavoidably introduced. The temptation to replace the terms *complex quantity*, *imaginary quantity* and *real quantity* by some such terms as *gonion*, *orthogon* and *agon* has been successfully resisted.

Partly in order to aid the student in obtaining a comparative view of the subject, partly in order to indicate in some detail the sources of information and give due credit to other writers, numerous foot-note references have been introduced.

I take great pleasure in acknowledging my obligations to Professor Haskell for valuable criticism and suggestion.

IRVING STRINGHAM.

UNIVERSITY OF CALIFORNIA,
BERKELEY, July 1, 1893.

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SIGNS AND ABBREVIATIONS.

The following quantitive and operational signs are used in this work and are collected here for convenience of reference.

$+$	Plus.
$-$	Minus.
$\sim$	Difference between.
$\pm$	Plus or minus.
$\times$	Multiplied by.
$/$	Divided by.
$\times\!/$	Multiplied or divided by.
$=$	Is equal to.
$\doteq$	Approaches as a limit.
$>$	Is greater than.
$<$	Is less than.
$\geqq$	Is not less than.
$\leqq$	Is not greater than.
$(>)$	Something greater than.
$(<)$	Something less than.
tsr	Tensor of (absolute value of).
vsr	Versor of.
amp	Amplitude of (argument of).
ln	Natural logarithm of.
e	Natural base.
π	Ratio of circumference to diameter.
$^{b}\log$	Logarithm of, to base b.
$\log_{\kappa}$	Logarithm of, to modulus κ.
$\sin_{\kappa}$, $\cos_{\kappa}$, etc.	Sine, cosine, etc., to modulus κ.
$\operatorname{limit}_{x \doteq a}$	Limit of, when x approaches a.
$\therefore$	Therefore.

UNIPLANAR ALGEBRA

INTRODUCTION.

EUCLID'S DOCTRINE OF PROPORTION.

A.—Notation.

In Sections B and C of this Introduction capital letters denote magnitudes, and when the pairs of magnitudes compared are both of the same kind they are denoted by letters taken from the early part of the alphabet, as A, B compared with C, D; but when they are, or may be, of different kinds, by letters taken from different parts of the alphabet, as A, B compared with P, Q or X, Y. Small italic letters m, n, p, q denote integers. By m. A or mA is meant the mth multiple of A, and it may be read m times A; by mn is meant the arithmetical product of the integers m and n, and it is assumed that $mn = nm$. The combination m. nA denotes the mth multiple of the nth multiple of A, and it is assumed that m. nA $= n$. mA $= mn$. A.

B.—Definitions and Axioms.*

1. "A greater magnitude is said to be a *multiple* of a less, when the greater contains the less an exact number of times."

2. "A less magnitude is said to be a *submultiple* of a greater, when the less is contained an exact number of times in the greater."

---

* The quoted paragraphs of Section B are transcribed in part from the *Syllabus of Plane Geometry*, published by the Association for the Improvement of Geometrical Teaching, in part from Hall and Stevens' *Text Book of Euclid's Elements*, Book V.

The following property of multiples is assumed as axiomatic:

(i). $m\,A > = \text{or} < m\,B$ according as $A > = \text{or} < B$. (Euc. Axioms 1 and 3.)

The converse necessarily follows:*

(ii). $A > = \text{or} < B$ according as $m\,A > = \text{or} < m\,B$. (Euc. Axioms 2 and 4.)

3. "The *ratio* of one magnitude to another of the same kind is the relation which the first bears to the second in respect of *quantuplicity*."

"The ratio of A to B is denoted thus, A : B; and A is called the *antecedent*, B the *consequent* of the ratio."

"The quantuplicity of A with respect to B may be estimated by examining how the multiples of A are distributed among the multiples of B, when both are arranged in ascending order of magnitude and the series of multiples continued without limit." This distribution may be represented graphically thus:

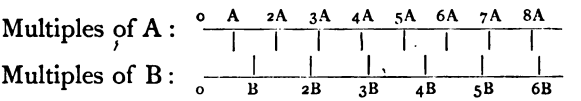


Fig. 1.

4. If, in this comparison of the multiples of two magnitudes, any multiple, as $n\,A$, of the one coincides with (is equal to) any multiple, as $m\,B$, of the other, the two magnitudes bear the same ratio to one another as the two numbers m, n, and are said to be *commensurable*, but

* "RULE OF CONVERSION. If the hypotheses of a group of demonstrated theorems be exhaustive (that is, form a set of alternatives of which one must be true), and if the conclusions be mutually exclusive (that is, be such that no two of them can be true at the same time), then the converse of every theorem of the group will necessarily be true." (*Syllabus of Plane Geometry*, p. 5.)

incommensurable if no such coincidence takes place, however far the process of comparison is carried.

5. The ratio of two magnitudes is said to be equal to a second ratio of two other magnitudes (whether of the same or of a different kind from the former), when the multiples of the antecedent of the first ratio are distributed among those of its consequent in the same order as the multiples of the antecedent of the second ratio among those of its consequent.

As tests of the equality of two ratios either of the following criteria may be employed, m and n being integers:

(i). The ratio of A to B is equal to that of P to Q,

when $\quad m\,\mathrm{A} > = \text{ or } < n\,\mathrm{B}$

according as $\quad m\,\mathrm{P} > = \text{ or } < n\,\mathrm{Q}.$

(ii). If m be any integer whatever and n another integer so determined that either $m\,\mathrm{A}$ is between $n\,\mathrm{B}$ and $(n+1)\,\mathrm{B}$ or is equal to $n\,\mathrm{B}$, then the ratio of A to B is equal to that of P to Q,

when $\quad m\mathrm{P}$ is between $n\mathrm{Q}$ and $(n+1)\mathrm{Q}$ or equal to $n\mathrm{Q}$

according as $m\mathrm{A}$ is between $n\mathrm{B}$ and $(n+1)\mathrm{B}$ or equal to $n\mathrm{B}$.

(iii). It should be remarked that the *rule of identity** is applicable to this definition, and that therefore, if the ratio of A to B be equal to that of P to Q,

then also $\quad m\,\mathrm{A} > = \text{ or } < n\,\mathrm{B}$

according as $\quad m\,\mathrm{P} > = \text{ or } < n\,\mathrm{Q}.$

6. "When the ratio of A to B is equal to that of P to Q, the four magnitudes are said to be *proportionals*, or to form a *proportion*. The proportion is denoted thus:

$$\mathrm{A} : \mathrm{B} :: \mathrm{P} : \mathrm{Q},$$

* RULE OF IDENTITY. If there be only one fact or state of things X, and only one fact or state of things Y, then from X is Y the converse Y is X of necessity follows.

which is read: A is to B as P is to Q. A and Q are called the *extremes*, B and P the *means*."

The proportion A : B :: P : Q may be represented graphically thus:

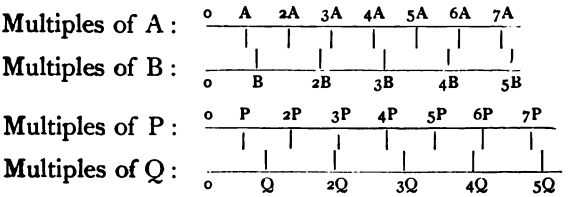


Fig. 2.

In a diagram of this kind it is obvious that in general, of the two figures thus compared, representing two equal ratios, one will be an enlarged copy of the other, but in particular, if the antecedents and consequents be respectively equal to one another, the two figures will be congruent.

7. "The ratio of one magnitude to another is greater than that of a third magnitude to a fourth, when it is possible to find equimultiples of the antecedents and equimultiples of the consequents such that while the multiple of the antecedent of the first ratio is greater than, or equal to, that of its consequent, the multiple of the antecedent of the second is not greater, or is less, than that of its consequent." That is,

$$A : B > P : Q,$$

if integers m, n can be found such that

if $\quad m\,A > n\,B$, then $m\,P \leqq n\,Q$,
or if $\quad m\,A = n\,B$, then $m\,P < n\,Q$.

8. "If A is equal to B, the ratio of A to B is called a *ratio of equality*.

"If A is greater than B, the ratio of A to B is called a *ratio of greater inequality*.

"If A is less than B, the ratio of A to B is called a *ratio of less inequality*."

9. "Two ratios are said to be *reciprocal* when the antecedent and consequent of one are the consequent and antecedent of the other respectively."

10. "Three magnitudes of the same kind are said to be *proportionals* when the ratio of the first to the second is equal to that of the second to the third."

11. "Three or more magnitudes are said to be in *continued proportion* when the ratio of the first to the second is equal to that of the second to the third, and the ratio of the second to the third is equal to that of the third to the fourth, and so on."

C.—Paraphrase from the Fifth Book of Euclid.*

Proposition 1.

"*Ratios which are equal to the same ratio are equal to one another.*"

Let A : B :: P : Q and X : Y :: P : Q, then A : B :: X : Y.

For, the multiples of each of the six magnitudes being ranged beside each other in pairs, as represented in Def. 6, it is obvious that in each case the multiples of the antecedent are distributed among those of the corresponding consequent in exactly the same order.

* Enunciations and numbering quoted from Hall and Stevens' *Text Book of Euclid's Elements*, Book V.

PROPOSITION 2.

"*If two ratios be equal, the antecedent of the second is greater than, equal to, or less than, its consequent, according as the antecedent of the first is greater than, equal to, or less than, its consequent.*"

Let $A : B :: P : Q$;
then $P > = \text{or} < Q$ according as $A > = \text{or} < B$.

This follows from Def. 5 (iii) by taking $m = n = 1$.

PROPOSITION 3.

"*If two ratios be equal, their reciprocal ratios are equal.*" (Invertendo.)

Let $A : B :: P : Q$, then $B : A :: Q : P$.

This is made evident graphically by constructing diagrams for the two ratios, as in Def. 6, and reading them off as a proportion, once in direct order (downwards) for the hypothesis, and again in inverse order (upwards) for the conclusion, and applying the rule of identity.

PROPOSITION 4.

"*Equal magnitudes have the same ratio to the same magnitude; and the same magnitude has the same ratio to equal magnitudes.*"

Let A, B, C be three magnitudes of the same kind, and let $A = B$; then $A : C :: B : C$ and $C : A :: C : B$.

For if $A = B$, their multiples are identical, and are therefore distributed in the same order among those of C.

$\therefore A : C :: B : C$, (Def. 5.)
and invertendo $C : A :: C : B$. (Prop. 3.)

PROPOSITION 5.

"Of two unequal magnitudes, the greater has a greater ratio to a third magnitude than the less has; and the same magnitude has a greater ratio to the less of two magnitudes than it has to the greater."

Let A, B, C be three magnitudes of the same kind;

then if $A > B$, so is $A : C > B : C$,
and if $B < A$, so is $C : B > C : A$.

First: If $A > B$, an integer m can be found such that mA exceeds mB by a magnitude greater than C. Hence, the integer n being so chosen that mA is equal to or greater than nC and less than $(n+1)C$, the conditions require that

$$mB < nC,$$

and therefore $A : C > B : C.$ (Def. 7.)

Second: If $B < A$, then taking m and n as in the foregoing proof the same inequalities between the multiples of A, B and C still exist, and because $nC > mB$ while nC is either $< mA$, or at most $= mA$,

$$\therefore\ C : B > C : A.$$ (Def. 7.)

PROPOSITION 6.

"Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another."

That is, A, B, C being three magnitudes of the same kind;

if $A : C :: B : C$, then $A = B$.
and if $C : A :: C : B$, then $A = B$.

The proof of this proposition is made a part of the proof of Prop. 7.

PROPOSITION 7.

"*That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.*"

That is, A, B, C being three magnitudes of the same kind;

if $A : C > B : C$, then $A > B$.

and if $C : A > C : B$, then $A < B$.

Proof of Propositions 6 and 7.

First Part: It has been proved that

$A : C :: B : C$, if $A = B$, (Prop. 4.)

and $A : C > B : C$, if $A > B$, (Prop. 5.)

and $A : C < B : C$, if $A < B$. (Prop. 5.)

Hence, by the *rule of conversion*, (Def. 2, Ax. ii.)

$$A > = \text{or} < B$$

according as $A : C > = \text{or} < B : C$.

This proves the first part of each of the two propositions.

Second Part: It has been proved that

$C : A :: C : B$, if $A = B$, (Prop. 4.)

and $C : A < C : B$, if $A > B$, (Prop. 5.)

and $C : A > C : B$, if $A < B$. (Prop. 5.)

Hence, by the *rule of conversion*, (Def. 2, Ax. ii.)

$$A > = \text{or} < B$$

according as $C : A < = \text{or} > C : B$.

This proves the second part of each of the two propositions.

(i). COROLLARY: *If* $A : C > B : C$, *then* $C : A < C : B$, *and conversely.*

(ii). COROLLARY: *If* $A : C > P : R$, *then* $C : A < R : P$, *and conversely.*

PROPOSITION 8.

"Magnitudes have the same ratio one to another which their equimultiples have."

Let A, B be two magnitudes of the same kind and m any integer; then

$$A : B :: mA : mB.$$

For if p, q be any two integers,

$$m.pA > = \text{or} < m.qB$$

according as $pA > = \text{or} < qB$. (Def. 2, Ax. ii.)

But $m.pA = p.mA$ and $m.qB = q.mB$;

$$\therefore\ p.mA > = \text{or} < q.mB$$

according as $pA > = \text{or} < qB$,

whatever integers p and q represent. Hence

$$A : B :: mA : mB.$$

(i). COROLLARY: *If* $A : B :: P : Q$, *then* $mA : mB :: nP : nQ$, *whatever integers m and n may be.*

PROPOSITION 9.

"If two ratios be equal, and any equimultiples of the antecedents and also of the consequents be taken, the multiple of the first antecedent has to that of its consequent the same ratio as the multiple of the other antecedent has to that of its consequent."

Let $A : B :: P : Q$, then $mA : nB :: mP : nQ$, m and n being any integers.

For, if p, q be any two integers, then, since by hypothesis

$$A : B :: P : Q,$$

$$\therefore\ pm.P > = \text{or} < qn.Q$$

according as $pm.A > = \text{or} < qn.B$; (Def. 5.)

that is, $p.mP > = \text{or} < q.nQ$

according as $p.mA > = \text{or} < q.nB$.

$$\therefore\ mA : nB :: mP : nQ.$$ (Def. 5.)

PROPOSITION 10.

"*If four magnitudes of the same kind be proportionals, the first is greater than, equal to, or less than, the third, according as the second is greater than, equal to, or less than, the fourth.*"

Let $A : B :: C : D$;
then $A > = \text{or} < C$ according as $B > = \text{or} < D$.

If $B > D$, then

	$A : B < A : D$;	(Prop. 5.)
but	$A : B :: C : D$,	(Hypoth.)
	$\therefore C : D < A : D$,	
	$\therefore A : D > C : D$,	
	$\therefore A > C$.	(Prop. 7.)

If $B = D$, then

	$C : B :: C : D$;	(Prop. 4.)
but	$A : B :: C : D$,	(Hypoth.)
	$\therefore A : B :: C : B$,	(Prop. 1.)
	$\therefore A = C$.	(Prop. 6.)

If $B < D$, then

	$A : B > A : D$;	(Prop. 5.)
but	$A : B :: C : D$,	(Hypoth.)
	$\therefore C : D > A : D$,	
	$\therefore C > A$ and $A < C$.	(Prop. 7.)

PROPOSITION 11.

"*If four magnitudes of the same kind be proportionals, the first will have to the third the same ratio as the second to the fourth.*" (Alternando.)

Let $A : B :: C : D$, then will $A : C :: B : D$.

For m and n being integers,

$$A : B :: mA : mB$$

and $$C : D :: nC : nD, \qquad \text{(Prop. 8.)}$$

$$\therefore mA : mB :: nC : nD \qquad \text{(Prop. 1.)}$$

$\therefore mA > = \text{or} < nC$ according as $mB > = \text{or} < nD$. (Prop. 10.)

But m and n are any integers;

$$\therefore A : C :: B : D. \qquad \text{(Def. 5.)}$$

D.—Seven Fundamental Theorems in Proportion.*

Proposition 12: (Lemma).**

"*If on two straight lines*, AB, CD, *cut by two parallel straight lines* AC, BD, *equimultiples of the intercepts respectively be taken; then the line joining the points of division will be parallel to* AC, *or* BD."

On AB and CD, produced either way, let the respective equimultiples BE, DF of AB, CD be taken, on the same side of BD; then EF is parallel to BD.

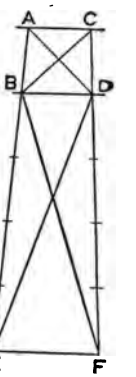


Fig. 3

For, join AD, DE, BC, BF. Since the triangles ABD, CBD are on the same base BD, and their vertices A, C are in the line AC parallel to BD, they are equal in area; and whatever multiple BE is of AB, or DF of CD, the triangle DBE is that same multiple of the triangle ABD, and the triangle DBF of the triangle CBD.

$\therefore$ area of triangle EBD = area of triangle FBD.

But these triangles EBD, FBD have the same base BD; hence their vertices E and F must be in a straight line parallel to BD and therefore EF is parallel to BD.

---

* Enunciations of Propositions 13-17 quoted from the *Syllabus of Plane Geometry*, Book IV, Section 2.

** J. M. Wilson: *Elementary Geometry*, page 205.

PROPOSITION 13.

"*If two straight lines be cut by three parallel straight lines, the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.*"

Let the three parallel straight lines A A′, B B′, C C′ be cut by two other straight lines A C, A′ C′ in the points A, B, C and A′, B′, C′ respectively; then

$$AB : BC :: A'B' : B'C'.$$

For, on A C take B M $= m$. A B, B N $= n$. B C, m and n being integers, M and N on the same side of B. Also on A′ C′ take B′ M′ $= m$. A′ B′, B′ N′ $= n$. B′ C′, M′ and N′ being on the same side of B′ as M and N of B. Then, by the foregoing lemma (Prop. 12), M M′ and N N′ are both parallel to B B′ and cannot meet. Hence, whatever integers m and n may represent,

B′ M′ (or m. A′ B′) $>$ = or $<$ B′ N′ (or n. B′ C′)

according as

B M (or m. A B) $>$ = or $<$ B N (or n. B C);

$\therefore$ A B : B C :: A′ B′ : B′ C′. (Def. 5).

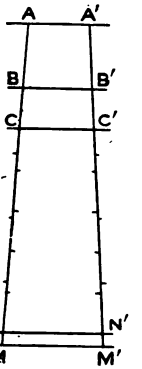


Fig. 4

(i). COROLLARY: "*If the sides of a triangle be cut by a straight line parallel to the base, the segments of one side are to one another in the same ratio as the segments of the other side.*"

(ii). COROLLARY: "*If two straight lines be cut by four or more parallel straight lines, the intercepts on the one are to one another in the same ratio as the corresponding intercepts on the other.*"

(iii). COROLLARY: *If in any triangle, as* O A B, *a straight line* E F, *parallel to the side* A B, *cut the other sides,* O A *in* E *and* O B *in* F, *then*

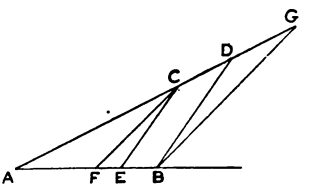
$$AB : EF :: OA : OE :: OB : OF.$$

PROPOSITION 14.

"A given finite straight line can be divided internally into segments having any given ratio, and also externally into segments having any given ratio except the ratio of equality;" and if the line be given in both length and sense, there is in each case one and only one such point of division.

Let A B be the given straight line; it may be divided, as at E, in a given ratio P : Q.

For, on the straight line A G making any convenient angle with A B lay off A C = P, C D = Q. Then C E, drawn parallel to D B to meet A B in E, will divide A B at E in the given ratio. (Prop. 13.)



Since C E and D B are parallel, C and E lie on the same side of D and B, and hence the division will be internal if A and D are on opposite sides of C, but external if A and D are on the same side of C.

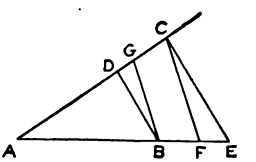


Fig. 5

If the line to be divided be estimated in a given sense, as from A to B, there is in each case only one point of division in the given ratio. For if any other point, as F, be joined to C and B G be drawn parallel to F C, then

A F : F B :: A C : C G, (Prop. 13.)

so that F divides A B in the ratio A C : C G, different from the given ratio.

If the given ratio be a ratio of equality, the construction in the case of external division fails.

PROPOSITION 15.

"A straight line which divides the sides of a triangle proportionally is parallel to the base of the triangle."

Let DE divide the sides AB, AC of the triangle ABC proportionally, so that

$$AD : DB :: AE : EC,$$

then DE is parallel to BC.

If possible, let DF be parallel to BC, F some other point than E; then

$$AD : DB :: AF : FC.$$ (Prop. 13.)

But by hypothesis

$$AD : DB :: AE : EC$$

$$\therefore\ AF : FC :: AE : EC$$

which is only possible when F coincides with E.

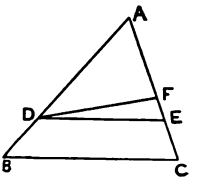


Fig. 6.

PROPOSITION 16.

"*Rectangles of equal altitude are to one another in the same ratio as their bases.*"

Let KA, KB be two rectangles having the common altitude OK and their bases OA, OB extending in the same line from O to the right; then

$$\text{rect. } KA : \text{rect. } KB :: OA : OB.$$

In the line OAB produced take

$OM = m.OA$, $ON = n.OB$,

m and n being integers, and complete the rectangles KM, KN. Whatever multiples OM and ON are of OA and OB, the rectangles KM and KN are the same respective multiples of the rectangles KA and KB; that is,

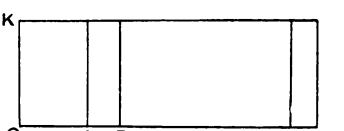


Fig. 7.

$$KM = m.KA,\ KN = n.KB,$$

and according as

$$OM \text{ (or } m.OA) > = \text{ or } < ON \text{ (or } m.OB)$$

so is KM (or $m.KA$) $> = \text{ or } <$ KN (or $n.KB$)

$\therefore$ rect. KA : rect. KB :: OA : OB. (Def. 5.)

(i). COROLLARY: "*Parallelograms or triangles of the same altitude are to one another as their bases.*"

PROPOSITION 17.

"*In the same circle, or in equal circles, angles at the centre and sectors are to one another as the arcs on which they stand.*"

Let there be two equal circles with centres at K and K', and on their circumferences any two arcs OA, O'B; then

angle OKA : angle $O'K'B$ :: OA : $O'B$,

and sector OKA : sector $O'K'B$ :: OA : $O'B$.

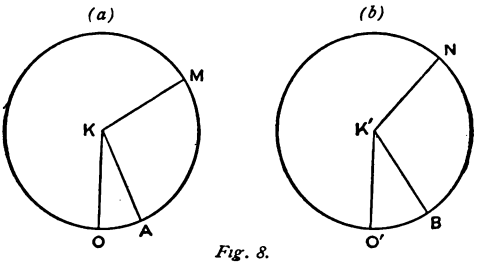


Fig. 8.

On the two circumferences respectively take

$$OM = m.OA, \quad O'N = n.O'B,$$

m and n being integers. Whatever multiples OM and $O'N$ are of OA and $O'B$, the same multiples respectively are the angles or sectors OKM and $O'K'N$ of the angles or sectors OKA and $O'K'B$; that is,

$$OKM = m.OKA, \quad O'K'N = n.O'K'B,$$

and according as

$$OM \text{ (or } m.OA) > = \text{ or } < O'N \text{ (or } n.O'B)$$

so is

$$OKM \text{ (or } m.OKA) > = \text{ or } < O'K'N \text{ (or } n.O'K'B).$$

$$\therefore\ OKA : O'K'B :: OA : O'B, \qquad \text{(Def. 5.)}$$

wherein OKA and O'K'B represent either angles or sectors.

(i). COROLLARY: *In any two given concentric circles, corresponding arcs intercepted by common radii bear always the same ratio to one another.*

That is, if u, u', u'', . . . be arcs on one of the circles determined by a series of radii, and the same radii intercept on the other circle the corresponding arcs v, v', v'', . . . then

$$u : v :: u' : v' :: u'' : v'' \ldots$$

PROPOSITION 18.

Arcs of circles that subtend the same angle or equal angles at their centres are to one another as their radii.

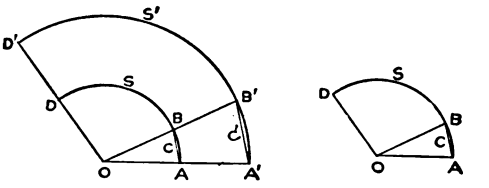


Fig. 9.

Let there be two arcs S = AD and S' = A'D', having their angles AOD and A'OD' either equal and distinct or common, and let R, R' be their respective radii; then

$$S : S' :: R : R'.$$

If the two arcs be not concentric, let them be made so, and let their bounding radii be made to coincide. Then

the proposition proved for the concentric will also be true for the non-concentric arcs.

Conceive the angle at O to be divided into m equal parts, m being any integer, by radii setting off the arcs S and S′ into the same number of equal parts, and draw the equal chords of the submultiple arcs of S and the like equal chords of the submultiple arcs of S′. Let C and C′ be the respective lengths of these chords.

Then, since the chords C, C′ cut off equal segments on the lines O A′, O B′ they are parallel (Prop. 12), and

$$C : C' :: R : R'. \quad \text{(Prop. 13, Cor. iii.)}$$

Therefore, m being any integer,

$$mC : mC' :: R : R'. \quad \text{(Prop. 8.)}$$

Let m be the number of equal parts into which the angle at O is divided; then mC and mC' are the lengths of the polygonal lines formed by the equal chords of S and S′ respectively.

If now m be increased indefinitely, the chords decrease in length but increase in number, and the two polygonal lines which they form approach coincidence with the arcs S and S′ respectively; and by increasing m sufficiently the aggregate of all the spaces between the arcs and their chords may be made smaller than any previously assigned arbitrarily small magnitude. Under these circumstances it is assumed as axiomatic that the relation existing between the polygonal lines exists also between the arcs, which are called *limits*. Under this assumption it follows that

$$S : S' :: R : R'. \quad \text{Q. E. D.}$$

(i). COROLLARY: *Circumferences are to one another as their radii.*

(ii). COROLLARY: *Of two arcs of circles that subtend the same angle or equal angles at their centres, that is the longer which has the longer radius.* (By Prop. 2.)

E.—AGENDA: SUPPLEMENTARY PROPOSITIONS.

(1). If two geometrical magnitudes A, B, have the same ratio as two integers m, n, prove that

$$n\,\mathrm{A} = m\,\mathrm{B}.$$

(2). If A, B be two geometrical magnitudes and m, n two integers such that $n\,\mathrm{A} = m\,\mathrm{B}$, prove that

$$\mathrm{A} : \mathrm{B} :: m : n.$$

Hence infer the statement in the first part of Definition 4, page 4, concerning commensurable magnitudes.

(3). Given A : B :: P : Q and $n\,\mathrm{A} = m\,\mathrm{B}$, prove that

$$n\,\mathrm{P} = m\,\mathrm{Q}.$$

(4). It is a corollary of (3), that if A : B :: P : Q and A be a multiple, part, or multiple of a part of B, then P is the same multiple, part, or multiple of a part of Q.

(5). Given A : B :: P : Q and B : C :: Q : R, prove that P $>$ = or $<$ R according as A $>$ = or $<$ C.

(6). Given A : B :: P : Q and B : C :: Q : R, prove that

$$\mathrm{A} : \mathrm{C} :: \mathrm{P} : \mathrm{R}.$$ *(Ex aequali.)*

(7). Given A : B :: P : Q and B : C :: Q : R and C : D :: R : S and D : E :: S : T, prove that

$$\mathrm{A} : \mathrm{E} :: \mathrm{P} : \mathrm{T}.$$

State and prove the general theorem of which this is a particular case.

(8). Given A : B :: Q : R and B : C :: P : Q, prove that

$$\mathrm{A} : \mathrm{C} :: \mathrm{P} : \mathrm{R}.$$

CHAPTER I.

LAWS OF ALGEBRAIC OPERATION.

I. QUANTITY.

1. Quantities in General. Quantities, whatever their nature, may be expressed in terms of geometrical magnitudes; in particular they may be thought of as straight lines of definite fixed or variable length. Such magnitudes, in so far as they represent the quantities of ordinary algebra, are of three kinds: real, imaginary, and complex; real if, when considered by themselves (laid off upon the real axis), they are supposed to involve only the idea of length, positive or negative, without regard to direction, imaginary when they involve not only length, but also turning or rotation through a right angle, that is, length and direction at right angles to the axis of real quantities, finally complex if they embody length and rotation through *any* angle, that is, length and unrestricted direction in the plane.

If we think of the straight line as generated by the motion of a point, we may translate *length positive or negative* into *motion forwards or backwards;* and it will sometimes be convenient to use the latter terminology in place of the former.

It is at once evident that both reals and imaginaries are particular forms of complex quantities, reals involving motion forwards or backwards and rotation through a zero-angle, imaginaries involving motion forwards or

backwards and rotation through a right angle. The three kinds of quantities will be considered in order; the distinction between them, here roughly outlined, will be made clearer by a study of their properties.

2. Nature of Real Quantities. It is evident that all real quantities may be made concretely cognizable by laying them off (in the imagination) as lengths, in the positive or negative sense, upon *one* straight line. In this representation every straight line suffices to embody in itself all real quantities, having its own positive and negative sense, that is, its direction forwards and backwards. In particular, all the numbers of common arithmetic, both integral and fractional, are accurately represented by distances laid off from a fixed origin in the positive sense upon a straight line, and in the same way all so-called irrational numbers, though only approximately realizable as true numbers in arithmetic, are accurately represented. Hence the following proposition, which is postulated as self-evident:

The laws of algebraic operation that obtain with geometrical real magnitudes, that is, lengths laid off upon a straight line, are IPSO FACTO *true when applied to arithmetical quantities, or numbers.*

But the converse of this proposition is not equally self-evident. For inasmuch as so-called irrational number, that is, quantity in general, is *not* realizable as true number (integer or fraction) in arithmetic, the proof that the laws of algebraic operation obtain for integers and fractions constitutes not a proof, but only a presumption, that they obtain also for so-called irrational number.

In the following pages magnitudes will be represented by straight lines of finite length.

II. Definitions of Algebraic Operations.

3. Algebraic Addition. *Simple addition* is here defined as the putting together, end to end, different line-segments, or links, in such a way as to form a one dimensional continuum, that is, a continuous straight line. This kind of addition corresponds to the addition of positive numbers in arithmetic.

Algebraic addition takes account of negative magnitudes, that is, of lines taken in the negative sense (from right to left, if positive lines extend from left to right), and to add to any line-segment a negative magnitude is to cut off from its positive extremity a portion equal in length to the negative magnitude. This kind of addition includes the addition and subtraction of positive numbers in arithmetic and introduces the new rule that larger positive magnitudes may be subtracted from smaller, producing thereby negative magnitudes. We then extend the idea of negativeness also to number and produce negative number, prefixing the sign — to positive number as a mark of the new quality.

The result of adding together algebraically several magnitudes is called a *sum*. In a sum the constituent parts are *terms*.

4. Zero is defined as the sum of a positive and an equally large negative magnitude; in symbols,

$$+a-a=0.$$

It is not a magnitude but indicates the absence of magnitude.

5. Algebraic Multiplication. On two straight lines making any convenient angle with one another at O, Fig. 10, lay off $OA=a$, $OB=b$, and on OA in the same

direction as OA lay off $OJ = j$, which shall be of fixed length in all constructions belonging to algebra and shall be called the *real unit*. Join J and B and draw from A a straight line parallel to JB to intersect OB in M. Then by Proposition 13 (p. 14) the intercepts on OA, OM by the parallels JB, AM are proportionals, and if $OM = m$,

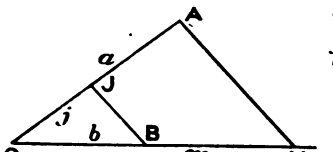


Fig. 10.

$$j : a :: b : m.$$

The length m, thus determined, is defined as the *algebraic product*, or simply the *product*, of the real magnitude a by the real magnitude b, and is denoted by $a \times b$, or by $a \cdot b$, or more simply still by ab.* In a product the constituent parts are *factors*.

The product $a \times b$ may also be a factor in another product, consisting therefore of three factors, as $(a \times b) \times c$, and this may in turn be a factor in a product of four factors, and so on.

6. Reciprocals. If m be equal to j, then

$$j : a :: b : j$$

and b, a, are called the *reciprocals* of a, b, respectively. These reciprocals are written thus:

j / a = reciprocal of a,
j / b = reciprocal of b,

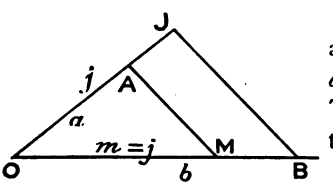


Fig. 11.

etc.; or more simply, since j remains unchanged throughout all algebraic operations, they may be conveniently

* Descartes: *la Géométrie*, reprint of 1886, p. 2.

represented by $/a$, $/b$, etc. Since the means, in any proportion between like magnitudes, may be interchanged,

$$\text{if } b = /a, \text{ then also } a = /b,$$

and *vice versa.*

A reciprocal, being itself a line-magnitude, may enter a product as one of its factors.

7. Idemfactor: Real Unit. If in the proportion $j : a :: b : m$ we write $a = j$, that is, make AM in Fig. 10 coincide with JB, then

$$j : j :: b : m, \text{ that is, } m = b. \qquad \text{(Prop. 2.)}$$

But by the definition of a product $m = j \times b$;

$$\therefore j \times b = b,$$

and in particular

$$j \times j = j.$$

An operator which, like j, as factor in a product leaves the other part of the product unchanged, is called an *idemfactor.** This particular real idemfactor is what was defined in Art. 5 as the real unit. In arithmetic it is denoted by the numerical symbol 1.

8. Quotient. The product defined by the proportion $j : c :: /a : m$ is $c \times /a$ and is called the *quotient* of c by a. The sign $\times$ before $/$ may be omitted without ambiguity and this quotient be denoted by the simpler notation c/a, in which c is called the *dividend* and a the *divisor.*

The proportion $j : a :: b : j$ defines

$$b = /a, \; a = /b, \text{ and } a \times b = j;$$

hence

$$a \times b = a \times /a = a/a = j,$$

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*Benjamin Peirce: *Linear Associative Algebra* (1870), p. 16, or *American Journal of Mathematics*, Vol. IV (1881), p. 104.

and so for any magnitude whatever. Hence we may describe the real unit as the quotient of any real magnitude by itself.

The quotient c/a is also represented by $c \div a$, or by $\frac{c}{a}$. The latter notation will be frequently employed in the sequel.

9. Agenda. Problems in Construction.

(1). From the definitions of Arts. 5, 6 and 8 prove that the following construction for the quotient a/b is correct: On one of two straight lines, making any convenient angle with one another, lay off $OA = a$, $OB = b$, on the other $OJ = j$; join B and J and draw AM parallel to BJ to intersect OJ in M. OM is the quotient sought.

(2). Given $a \times a = m$, construct a.

(3). Given a, b and c, construct $(a \times b) \times c$.

(4). Prove that $a \times b$ is $> = $ or $< b$ according as a is $> = $ or < 1

(5). Draw OX and OY making any convenient angle with each other; on OY lay off $OJ = j$, $OA = a$, $OB = b$, and on OX take $OJ_1 = j$. A straight line through J parallel to OX will be cut by J_1A and J_1B in two points P and Q. Show that if A and B are on the same side of O, the distance between P and Q is $PQ = /a \sim /b$, where $\sim$ means difference between, but if A and B are on opposite sides of O, then $PQ = /a + /b$. In this construction a and b are supposed to be positive magnitudes.

10. Infinity is defined as the reciprocal of zero; in symbols

$$/0 = \infty.$$

When a magnitude decreases and becomes zero, its reciprocal obviously increases and becomes infinite. Since zero is not a magnitude, neither is infinity as here defined.

In Art. 6 it was shown that $b = / a$ implies also $a = / b$; hence, from the definition $/ 0 = \infty$ follows

$$/ \infty = 0.$$

The construction for a product (Art. 5) shows that when one of its factors becomes 0 or ∞, the other remaining finite, the product itself is also 0 or ∞, so that for all finite values of a

$$a / 0 = a \times \infty = \infty,$$
$$a / \infty = a \times 0 = 0.$$

From the definition of addition (Art. 3) it is also obvious that

$$\pm 0 \pm a = \pm a,$$
$$\pm \infty \pm a = \pm \infty.$$

11. Indeterminate Algebraic Forms. When a sum or product assumes one of the forms $+\infty - \infty$, $0 \times \infty$, $0/0$, ∞ / ∞, it is said to be *indeterminate*, by which is meant: the form by itself gives no information concerning its own value.

(i). *The form* $+ \infty - \infty$. On a straight line ABP take at random two points A, B, so that AB is any real finite magnitude whatever. Take P, Q, R, on the same line,

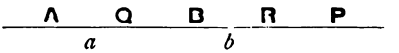

Fig. 12.

such that

$$AP = / AQ = / a, \quad BP = / BR = / b,$$

and let Q pass into coincidence with A. P then passes out

of finite range, R passes into coincidence with B and the difference

$$AB = /a - /b,$$

whatever its original value, assumes the form

$$/o - /o = \infty - \infty.$$

Hence, taken by itself, $\infty - \infty$ gives no information concerning its own value and is indeterminate.

(ii). *The forms* $o \times \infty$, o/o, ∞/∞. In the figure of Art. 5, let M and J remain fixed, while MA and JB, being always parallel to one another, turn about M and J until MA coincides with, and JB becomes parallel to OM. At this instant a becomes zero, b infinite, and $a \times b$ assumes the form $o \times \infty$; and because the original value of $a \times b$ is anything we choose to make it, the expression $o \times \infty$ gives no information concerning its own value and is therefore an indeterminate form.

Since $/o = \infty$ and $/\infty = o$, we may replace o/o by $o \times \infty$ and ∞/∞ by $\infty \times o$. The two forms o/o and ∞/∞ are therefore also indeterminate.

An expression, such as $/a - /b$, that gives rise to an indeterminate form, may nevertheless approach a determinate value as it nears its critical stage. To find this value is described as evaluating the indeterminate form. (See Arts. 43 and 52.)

III. Law of Signs for Real Quantities.

12. **In Addition and Subtraction.** The sign $+$, by definition, indicates that the magnitude following it is to be added algebraically to what precedes, without having its character as a magnitude in any way changed; and the sign $-$ indicates that the magnitude immediately

following it is to be reversed in sense (taken in the opposite direction) and then added algebraically to what precedes.

Any symbolic representative of quantity, a letter for example, unattended by either of the signs $+$ or $-$, but still thought of as part of an algebraic sum, is supposed to have the same relation and effect in such a sum as if it had before it the sign $+$. This usage necessitates the following law of signs in addition:

$$+ + = +, \quad + - = -,$$
$$- - = +, \quad - + = -;$$

thus

$$+ (+ a) = + a, \quad + (- a) = - a,$$
$$- (- a) = + a, \quad - (+ a) = - a,$$

and by a, unattended by any sign, is understood $+ a$.

13. In Multiplication and Division. In Art. 8 it was agreed that a / b shall stand for the product $a \times / b$. This convention requires that the combination $\times /$ shall produce $/$. Looked at from another point of view, the symbol $\times$ indicates that the letter following it is to be used as a factor with its character as a magnitude unchanged, while $/$ gives notice that the reciprocal of the magnitude immediately following it is to be used as a factor.

Any symbolic representative of quantity, a letter for example, unattended by either of the signs $\times$ or $/$, but still thought of as part of (factor in) an algebraic product, is supposed to have the same relation and effect in such product as if it had before it the sign $\times$. The usage here described necessitates the following law of signs for $\times$ and $/$:

$$\times \times = \times, \quad \times / = /,$$
$$/ / = \times, \quad / \times = /;$$

thus

$$\times (\times a) = \times a, \quad \times (/ a) = / a,$$
$$/ (/ a) = \times a, \quad / (\times a) = / a.$$

In practice the sign $\times$ is usually omitted, or replaced by a dot; thus: $a \times b = ab = a \cdot b$.

14. In Combination with each other: $+$, $-$ with $\times$, $/$. In the construction of a product any factor affected with the negative sign $-$ must be laid off in the sense opposite to the one it takes when affected with the positive sign $+$, and the constructions involving negative factors lead to the following rule:

An odd number of negative factors produces a negative product;

An even number of negative factors produces a positive product.

For, suppose one factor, as a, to be affected with the negative sign. The construction of $(-a) \times b$ is then as follows: Lay off b in the positive sense, say to the right, and OJ, the real unit, in the positive sense, say upwards: then $-a$ must extend downwards along JO produced. Join JB and draw AM parallel to BJ to intersect BO, produced backwards, in M. The product $(-a) \times b$ is thus the negative magnitude $-m$.

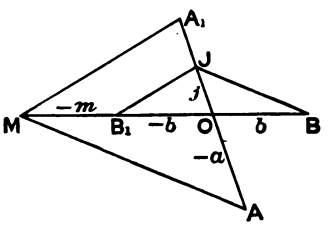

Fig. 13.

When the product is in the form $a \times (-b)$, $-b$ must be laid off in the negative sense towards M, $-b = OB_1$, a in

the positive sense towards A_1, $a = OA_1$, and it is easy to prove by proportion in the similar triangles thus formed, that the line through A_1 parallel to JB_1 intersects OB_1 in M. It is also obvious that $+a \times b = +m$. Hence

$$(-a) \times b = a \times (-b) = -a \times b.$$

If both factors are affected with the negative sign, the construction is as follows: Draw $OB = -b$ in the negative, $OA = -a$ in the negative, $OJ = j$ in the positive sense, AM parallel to BJ, intersecting BO produced in M. Then $OM = m$ is positive and by writing the proportions for the similar triangles OBJ and OMA it is easy to show that

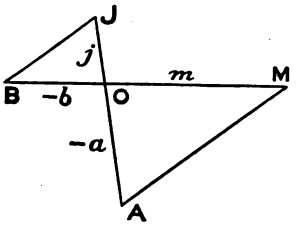

Fig. 14.

$$(-a) \times (-b) = +a \times b.$$

Thus the product of one negative factor and any number of positive factors is negative, while every pair of negative factors yields only positive products. Hence the proposition, Q. E. D.

IV. Associative Law for Real Quantities.

15. In Addition and Subtraction. The sequence of the terms of a sum remaining unchanged, the terms may be added separately, or in groups of two or more, indiscriminately, without disturbing the value of the sum ; that is,

$$(a + b) + c = a + (b + c) = a + b + c,$$

where a, b, c may represent positive or negative magnitudes indiscriminately. This is made evident at once by laying off and comparing with one another the lines $a + b$, c, and a, $b + c$, and a, b, c, taken in the proper sense and in the order indicated in the three groupings.

In introducing negative magnitudes the law of combinations of the signs + and −, as described in Art. 12, must be observed. The complete symbolical statement of the associative law for addition and subtraction is contained in the formula:

$$\pm(\pm a \pm b) = \pm(\pm a) \pm(\pm b),$$

wherein the order of occurrence of the signs + and − must be the same in the two members of the equation.

16. In Multiplication and Division. Construct upon OA and OM, Fig. 15, the products

$$OL = a \times b, \quad OM = a \times b, \quad OE = b \times c,$$
$$ON = (a \times b) \times c, \quad ON_1 = a \times (b \times c).$$

In this construction the lines marked f_1, f_2, f_3, through L, N and B, E and J, C are, by the rule for constructing products

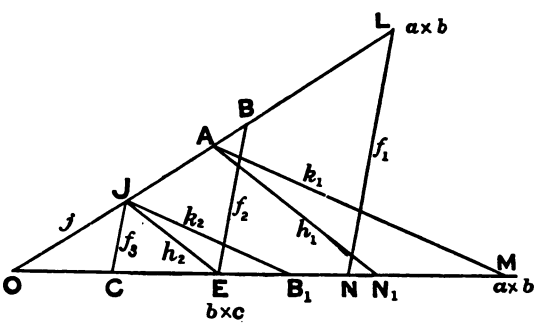

Fig. 15.

(Art. 5), parallel to each other, as are also the lines h_1, h_2 through A, N_1 and J, E, and the lines k_1, k_2 through A, M and J, B_1. Hence by proportion, as determined by the parallels f_2, f_1,

$$b : a \times b :: b \times c : (a \times b) \times c, \quad \text{(Prop. 13.)}$$

and as determined by the parallels k_2, k_1,

$$j : a :: b : a \times b, \qquad \text{(Prop. 13.)}$$
$$\therefore \quad j : a :: b \times c : (a \times b) \times c. \qquad \text{(Prop. 1.)}$$

But the parallels h_2, h_1 determine the proportion

$$j : a :: b \times c : a \times (b \times c).$$

Therefore N, N_1 are one and the same point (Prop. 6) and

$$\times (\times a \times b) \times c = \times a \times (\times b \times c).$$

Thus the sign $\times$ is distributive over the successive factors of a product; that is, as here follows when $a = j$,

$$\times (b \times c) = (\times b) \times (\times c).$$

The same is true of the sign /; for the product of any magnitude by its reciprocal is j (Art. 6), and two products that have their factors respectively equal are obviously themselves equal (Art. 5); that is,

$$/ a \times a = j, \quad b \times / b = j,$$
and $$(/ a \times a) \times (b \times / b) = j \times j = j. \qquad \text{(Art. 7.)}$$

Whence, by the associative law in multiplication just proved, and because the means in a proportion between like magnitudes may be interchanged (Prop. 11),

$$/ a \times (a \times b) \times / b = j,$$
and $$(/ a \times / b) \times (a \times b) = j;$$
$$\therefore \quad / (a \times b) = / a \times / b. \qquad \text{(Art. 6.)}$$

Thus the associative law for the sign / is the same as that for the sign $\times$, and its general statement is

$$\diagup (\diagup a \diagup b) = \diagup (\diagup a) \diagup (\diagup b),$$

wherein the order of occurrence of the signs $\times$ and / in the two members must be the same, and the law of their combination (Art. 13) must be observed.

V. Commutative Law for Real Quantities.

17. In Addition and Subtraction. Let a and b represent any two lengths taken in the same sense along a straight line such that $a = OA$ and $b = AB$; then

$$a + b = OB.$$

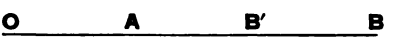

Fig. 16.

Let B' be so taken that $OB' = b$; then $B'B = a$. Therefore

$$a + b = OB = b + a.$$

If AB be negative and equal to $-b$, lay off a from O to the right, b from the extremity of a to the left. B will fall to the right or left of O according as b is less than or

Fig. 17.

greater than a; and in either case

$$a - b = OB.$$

In the former OB will be positive, in the latter negative. If now B' be so taken that $B'O = b$, then $B'B = a$. Hence

$$OB = -b + a = a - b.$$

If $a \pm b = c$ and d be any fourth magnitude, then

$$c \pm d = \pm d + c = a \pm b \pm d = \pm d \pm b + a.$$

Obviously any algebraic sum may have its terms commuted in like manner. Stated symbolically, this law is

$$\pm a \pm b = \pm b \pm a,$$

wherein each letter carries with it like signs on both sides of the equation.

18. In Multiplication and Division. On two straight lines, meeting at a convenient angle, construct OB, OB_1, each equal to b, and OA_1, OA, each equal to a. On OB_1 take $OJ = j$, the real unit, join JA_1, JB, draw AM, B_1N, parallel respectively to JB, JA_1, intersecting OB (produced, if necessary) in M and N, and let $OM = m$, $ON = n$. By definition of a product (Art. 5),

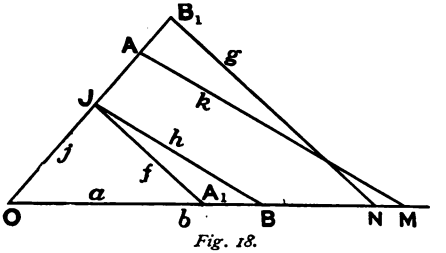

Fig. 18.

$$a \times b = m,$$

and $b \times a = n$.

But the proportion determined by the parallels h and k is

$$j : a :: b : m.$$ (Prop. 13.)

and that determined by the parallels f and g is

$$j : b :: a : n,$$

which by interchange of means (Prop. 11) becomes

$$j : a :: b : n.$$

$$\therefore \quad m = n.$$ (Prop. 6.)

M and N are hence one and the same point, and

$$b \times a = a \times b.$$

Inasmuch as the reciprocal of a line-magnitude is itself a line-magnitude, it follows from the above demonstration that

$$b \times / a = / a \times b;$$

or, observing the law of signs (Art. 13), this is

$$b / a = / a \times b;$$

and if b carry with it its proper factor sign, the more explicit form of statement contained in this equation is

$$\times b / a = / a \times b.$$

The formula for the commutative law in multiplication and division may therefore be written

$$\diagup b \diagup a = \diagup a \diagup b,$$

and this is easily extended to products of three or more factors. In this formula, the same sign attaches to a or b on both sides of the equation.

19. Agenda: Theorems in Proportion.

(1). Prove that in a proportion the product of the means is equal to the product of the extremes.

(2). Prove that rectangles are to one another in the same ratio as the products of their bases by their altitudes. Show that the same relation holds for parallelograms and triangles.

(3). If a square whose side is 1 be taken as the unit of area, the area of any rectangle (or parallelogram) is equal to the product of its base by its altitude.

VI. The Distributive Law for Real Quantities.

20. With the Sign of Multiplication. On OH and OK (Fig. 19), take $OC = c$, $OE = a + b$, and construct the product $(a + b) \times c = OG$. Then if OJ be the real unit, laid off on OK, EG and JC are parallel by construction. On OK take $OA = a$, and construct the product $a \times c = OF$. Then AF and EG, being parallel to JC, are parallel to one another, and FB drawn parallel to OK is equal to AE, that is, $FB = b$, and therefore FG is the product $b \times c$. Hence

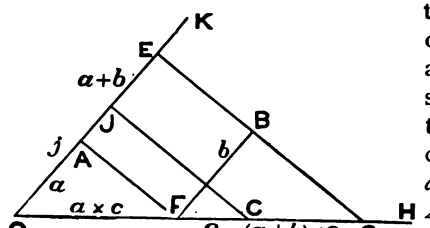

Fig. 19.

$$(a+b)\times c = a\times c + b\times c = c\times(a+b),$$

which is the law of distribution in multiplication and addition.

The construction and proof are not essentially different when some or all the magnitudes involved are affected with the negative instead of the positive sign.

The second factor may also consist of two or more terms; for

$$(a+b)\times(c+d) = a\times(c+d) + b\times(c+d),$$

by the distributive law,

$$= (c+d)\times a + (c+d)\times b,$$

by the commutative law,

and again, by the distributive and commutative laws,

$$(c+d)\times a + (c+d)\times b = c\times a + d\times a + c\times b + d\times b$$
$$= a\times c + a\times d + b\times c + b\times d.$$

Hence, replacing the sign $+$ by the double sign $\pm$, the completed formula for the distributive law in multiplication is

$$(\pm a \pm b)\times(\pm c \pm d) = (\pm a)\times(\pm c) + (\pm a)\times(\pm d)$$
$$+ (\pm b)\times(\pm c) + (\pm b)\times(\pm d).$$

21. **With the Sign of Reciprocation.** Since a, b, c, d, in the formula last written, are any real magnitudes whatever, they may be replaced by their reciprocals, $/a$, $/b$, $/c$, $/d$. For the same reason, in the formula

$$(\pm a \pm b)\times(\pm c) = (\pm a)\times(\pm c) + (\pm b)\times(\pm c),$$

$\times(\pm c)$ may be replaced by $/(\pm c)$, giving the corresponding distributive law with the sign of reciprocation:

$$(\pm a \pm b)/(\pm c) = (\pm a)/(\pm c) + (\pm b)/(\pm c).$$

But while the sign $\times$ is distributive over the successive terms of a sum (Art. 20), that is:

$$\times (\pm a \pm b \pm \ldots) = \times (\pm a) + \times (\pm b) + \times \ldots,$$

the sign / is not, as may be readily seen by constructing the product $c \times / (a + b)$, and the sum of products $c \times / a + c \times / b$, and comparing the results, which will be found to differ.

22. Agenda: Theorems in Proportion, Arithmetical Multiplication and Division.

(1). If $A : B :: C : D :: E : F :: \ldots,$
prove that

$$A : B :: A + C + E + \ldots : B + D + F + \ldots$$

(Addendo.)

(2). If $A : B :: C : D,$
prove that

$$A + B : B :: C + D : D \quad \textit{(Componendo.)}$$

and that

$$A \sim B : B :: C \sim D : D. \quad \textit{(Dividendo.)}$$

(3). If $A : B :: C : D,$
prove that

$$A + B : A \sim B :: C + D : C \sim D.$$

(4). If $A : B :: P : Q,$ $B : C :: Q : R,$
$C : D :: R : S,$ and $D : E :: S : T,$
prove that

$$A : E :: P : T. \quad \textit{(Ex æquali.)}$$

(5). If $A : B :: Q : R$ and $B : C :: P : Q,$
prove that

$$A : C :: P : R.$$

(6). Show that corresponding to

$$c \times (\times a + \times b) = \times (\times c \times a + \times c \times b)$$

there is the analogous formula

$$c\,/\,(/\,a + /\,b) = /\,(/\,c\,/\,a + /\,c\,/\,b).$$

(7). Construct the product 3×2 and show that the result is 6.

(8). Construct the quotient $1\,2\,/\,3$ and show that the result is 4.

(9). Construct $5\,/\,3$ and $2 \times 5\,/\,3$.

VII. Exponentials and Logarithms.

23. Definitions.* Suppose P, Q to be two points moving in a straight line, the former with a velocity, more strictly a *speed*,** proportional to its distance from a fixed origin O, the latter with a constant speed. Let x denote

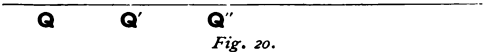

Fig. 20.

the variable distance of P, y that of Q from the origin, and let λ be the speed of P when $x = OJ = 1$, μ the constant speed of Q. As it arrives at the positions J, P', P'' successively, P is moving at the rates:

$$\lambda, \quad x'\lambda, \quad x''\lambda, \text{ respectively,}$$

the corresponding values of x and y are:

$$x = OJ = 1, \quad x' = 1 + JP', \quad x'' = 1 + JP'',$$
$$y = o, \qquad y' = OQ', \qquad y'' = OQ''$$

---

* Napier's definition of a logarithm. Napier: *Mirifici logarithmorum canonis descriptio* (Lond. 1620), Defs. 1-6, pp. 1-3, and the *Construction of the Wonderful Canon of Logarithms* (Macdonald's translation, 1889), Def. 26, p. 19. Also MacLaurin: *Treatise of Fluxions*, vol. i, chap. vi, p. 158, and Montucla: *Histoire des Mathematiques*, t. ii, pp. 16-17, 97.

** The speed of a moving point is the amount of its rate of change of position regardless of direction. Velocity takes account of change of direction as well as amount of motion. See Macgregor: *Elementary Treatise on Kinematics and Dynamics*, pp. 22, 23 and 55.

and Q is supposed to pass the origin at the instant when P passes J, at which point $x = OJ = 1$. The speed of P relatively to that of Q, or vice versa, is obviously known as soon as the ratio of μ to λ is given. By means of this construction the terms modulus, base, exponential, and logarithm are defined as follows:

(i). μ/λ, a given value of which, say m, determines a system of corresponding distances $x', x'', \ldots$ and $y', y'', \ldots$ is called the *modulus* of the system.

(ii). The modulus having been assigned, the value of x, corresponding to $y = OJ = 1$, is determined as a fixed magnitude, and is called the *base* of the system. Let it be denoted by b.

(iii). $x, x', x'', \ldots$ are called the *exponentials* of $y, y', y'', \ldots$ respectively, with reference either to the modulus m or the base b, and the relation between x and y is written

$$x \equiv \exp_m y, \quad \text{or } x \equiv b^y.**$$

(iv). $y, y', y'', \ldots$ are called the *logarithms* of $x, x', x'', \ldots$ respectively, with reference to the modulus m, or the base b, and the symbolic statement of this definition is either

$$y \equiv \log_m x, \quad \text{or } y \equiv {}^b\log x.*$$

The convention that P shall be at J when Q is at O introduces the convenient relations

$$\log_m 1 = 0, \quad \text{and } \exp_m 0 = 1, \text{ or } b^0 = 1,$$

and because (by definition) $y = 1$ when $x = b$,

$$\therefore \quad \log_m b = 1, \quad \text{and } \exp_m 1 = b, \text{ or } b^1 = b.$$

---

* ${}^b\log x$ is the German notation. English and American usage has hitherto favored writing the base as a subscript to log thus, $\log_b$; but Mr. Cathcart in his translation of Harnack's *Differential u. Integral Rechnung* has retained the German form, which is here adopted as preferable.

** See Appendix, page 139.

Exponentials and logarithms are said to be *inverse* to each other.

(v). The logarithms whose modulus is unity are called *natural* logarithms,* and the corresponding base is called *natural* base, the special symbols for which are ln and e respectively; thus

$$y = \log_1 x = \ln x, \quad \text{and } x = e^y = \exp y$$

represent the logarithm and its inverse, the exponential, in the natural system.

24. Relations between Base and Modulus. Let the speed of P remain equal to λx as in Art. 23 while the speed of Q is changed from μ to $k\mu$. The modulus, $\mu / \lambda = m$, will then be changed to $k\,\mu / \lambda = k\,m$, and the distance of Q from the origin, corresponding to the distance x, will become ky. Hence

$$ky = \log_{km} x = k \log_m x;$$

that is: *To multiply the modulus of a logarithm by any real quantity has the effect of multiplying the logarithm itself by the same quantity.*

In particular we may write

$$\log_m x = m \ln x.$$

Corresponding to this relation between logarithms in the systems whose moduli are m and km, the inverse, or exponential relation is

$$\exp_{km} ky = \exp_m y = \exp_m (\tfrac{1}{k}\, ky).$$

---

* These are also sometimes called Napierian logarithms, but it is well known that the numbers of Napier's original tables are not natural logarithms. The relation between them is expressed by the formula

$$\text{Napierian log of } x = 10^7 \ln (10^7 / x).$$

Napier's system was not defined with reference to a base or modulus. See the article on *Napier* by J. W. L. Glaisher in the *Encyclopædia Britannica*, ninth edition.

Let c be the base in the system whose modulus is km; then since $y = 1$ when $x = b$, and

$$ky = \log_{km} x = \log_{km} c = 1, \text{ when } x = c,$$
$$\therefore \quad b = \exp_{km} k = c^k.$$

and

$$c = \exp_m \tfrac{1}{k} = b^{1/k}.*$$

Hence: *If the modulus be changed from m to km, the corresponding base is changed from b to $b^{1/k}$.*

Again, if in the equation $\log_m x = m \ln x$, b be substituted for x, the value of m is obtained in the form

$$m = 1 / \ln b.$$

Hence in terms of the base of a system the logarithm is

$$\log_m x = \ln x / \ln b,$$

and if to x the value e be given, m is obtained in the form

$$m = \log_m e;$$

whence passing to the corresponding inverse relation:

$$\exp_m m = b^m = e;$$

that is: *The exponential of any quantity with respect to itself as a modulus is equal to natural base.*

25. The Law of Involution. By virtue of the fundamental principle of the last article—that to multiply the modulus multiplies the logarithm by the same amount—we have in general

$$\exp_{km} z = \exp_m z / k$$

and

$$\exp_{m/k} z = \exp_m kz.$$

Hence

$$\exp_m hk = \exp_{m/h} k = \exp_{m/k} h = \exp_{m/hk} 1,$$

*For an amplification of this proof see Appendix, page 139.

or in other terms, since changing m into m/h, or m/k, changes b into b^h, or b^k,

$$b^{hk} = (b^h)^k = (b^k)^h,$$

which expresses the law of *involution*. Obviously h, or k, or both, may be replaced by their reciprocals in this formula and the law, more completely stated, is

$$(b^{\times h})^{\times k} = (b^{\times k})^{\times h}.$$

Evolution. If k be an integer, the process indicated by $b^{1:k}$ is called evolution; and when $k = 2$, it is usually expressed by the notation $\sqrt{b}$.

26. The Law of Metathesis. Let $z = b^h$; then by the law of involution

$$z^k = (b^h)^k = b^{hk};$$

whence, passing to the corresponding inverse relations,

$$hk = \log_m z^k$$

and

$$h = \log_m z;$$

$$\therefore \quad k \log_m z = \log_m z^k,$$

which expresses the law of interchange of exponents with coefficients in logarithms,—the law of *metathesis*.

27. The Law of Indices.* The law of indices, or of addition of exponents, follows very simply from the definition of the exponential, thus: In the construction of Art. 23, and by virtue of Definition (iii) of that Article

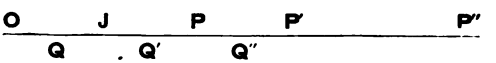

Fig. 20.

$$\exp_m y' = 1 + JP', \quad \exp_m y'' = 1 + JP''$$

$$\therefore \quad \exp_m y''/\exp_m y' = OP''/OP'$$

$$= 1 + P'P''/OP'.$$

* For an alternative proof see Appendix, pages 139-141.

But the magnitude $1 + P'P'' \; OP'$ represents the distance that P would traverse, starting at unit's distance to the left of P', on the supposition that its speed at P' is $\lambda x'/x' = \lambda$, and the corresponding distance passed over by Q is $y'' - y'$; hence, by definition of the exponential,

$$1 + P'P'' \; OP' = \exp_m (y'' - y'),$$

and therefore

$$\exp_m y'' / \exp_m y' = \exp_m (y'' - y'),$$

which is the law of *indices* for a quotient. In particular, if $y'' = 0$

$$\exp_m 0 \quad \exp_m y' = \exp_m (0 - y')$$

or

$$1 / \exp_m y' = \exp_m (-y'); \qquad \text{(Art. 23.)}$$

and therefore, writing $-y' = y$, the foregoing equation, $\exp_m y'' / \exp_m y' = \exp_m (y'' - y')$, becomes

$$\exp_m y'' \times \exp_m y = \exp_m (y'' + y),$$

which is the law of *indices* for a direct product. The complete statement of this law is therefore embodied in the formula

$$\exp_m y'' \times\!\!/ \exp_m y' = \exp_m (y'' \pm y'),$$

or its equivalent

$$b^{y''} \times\!\!/ b^{y'} = b^{y'' \pm y'}.$$

28. **The Addition Theorem.** Operating upon both sides of the equation last written with $\log_m$ we have

$$\log_m (b^{y''} \times\!\!/ b^{y'}) = y'' \pm y';$$

that is, replacing $b^{y''}$, $b^{y'}$ by x'', x' and y'', y' by $\log_m x''$, $\log_m x'$,

$$\log_m (x'' \times\!\!/ x') = \log_m x'' \pm \log_m x',$$

which is the *addition theorem* for logarithms.

29. Infinite Values of a Logarithm. If, in the construction of Art. 23, λ and μ become larger than any previously assigned arbitrarily large value, while their ratio m (that is, the modulus) remains unchanged, P and Q are transported instantly to an indefinitely great distance, and OP, OQ become simultaneously larger than any assignable magnitude. It is customary to express this fact in brief by writing

$$b^{\infty} = \infty, \quad \log_m \infty = \infty;$$

though to suppose these values actually attained would require both λ and μ to become actually infinite. This supposition will be justifiable whenever we find it legitimate, under the given conditions, to assign to the indeterminate form $\mu / \lambda = \infty / \infty$ a determinate value m.*

In like manner, since from $b^{\infty} = \infty$ we may infer $b^{-\infty} = 1 / \infty = 0$, the equations

$$b^{-\infty} = 0, \quad \log_m 0 = -\infty$$

are employed as conventional renderings of the fact, that when P and Q are moving to the left, P passing from J towards O and Q negatively away from O, x, in the equations $x = b^y$, $y = \log_m x$, remains positive and approaches 0, while y is negative and approaches $-\infty$.

30. Indeterminate Exponential Forms. When $v \times \log_m u$ becomes either $\pm 0 \times \infty$, or $\pm \infty \times 0$, it is indeterminate (Art. 11). Now $\log_m u$ is 0 if $u = 1$, is $+\infty$ if $u = +\infty$, and is $-\infty$ if $u = 0$ (Art. 29). Hence $v \times \log_m u$ will assume an indeterminate form under the following conditions:

* This form of statement must be regarded as conventional. Strictly speaking we cannot *assign a value* to an indeterminate form. When the quotient x / y, in approaching the indeterminate form, remains equal to, or tends to assume, a definitive value, we *substitute* this value for the quotient and call it a *limit*. In conventional language the indeterminate form is then said to be *evaluated*.

When $v = \infty$ and $u = 1$,
or $v = 0$ and $u = +\infty$,
or $v = 0$ and $u = 0$.

But if $v \times \log_m u$, or $\log_m u^v$ is indeterminate, so is u^v, and therefore the forms 1^∞, ∞°, 0° are indeterminate.

Whenever one of these forms presents itself, we write $y = u^v$ and, operating with ln, examine the form

$$\ln y = v \times \ln u.$$

If then $\ln y$ can be determined, y can be found through the equation

$$y = e^{\ln y}.$$

VIII. Synopsis of Laws of Algebraic Operation.*

31. Law of Signs:

(i). The concurrence of like signs gives the direct sign, $+$ or $\times$. Thus:

$$+ + = +, \quad - - = +, \quad \times \times = \times, \quad / / = \times.$$

(ii). The concurrence of unlike signs gives the inverse sign, $-$ or $/$. Thus:

$$+ - = -, \quad - + = -, \quad \times / = /, \quad / \times = /.$$

(iii). The concurrence of two or more positive, or an even number of negative factors, in a product or quotient, gives a positive result. Thus:

$$(+a) \diagup\!\!\!\diagup (+b) = +a \diagup\!\!\!\diagup b, \quad (-a) \diagup\!\!\!\diagup (-b) = +a \diagup\!\!\!\diagup b.$$

(iv). The concurrence of an odd number of negative factors, in a product or quotient, gives a negative result. Thus:

$$(+a) \diagup\!\!\!\diagup (-b) = -a \diagup\!\!\!\diagup b, \quad (-a) \diagup\!\!\!\diagup (+b) = -a \diagup\!\!\!\diagup b.$$

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*Cf. Chrystal: *Algebra*, vol. I, pp. 20–22.

32. Law of Association:

For addition and subtraction:

$$\pm(\pm a \pm b) = \pm(\pm a) \pm (\pm b).$$

For multiplication and division:

$$\diagup\!\!\diagdown(\diagup\!\!\diagdown a \diagup\!\!\diagdown b) = \diagup\!\!\diagdown(\diagup\!\!\diagdown a) \diagup\!\!\diagdown(\diagup\!\!\diagdown b).$$

33. Law of Commutation:

For addition and subtraction:

$$\pm a \pm b = \pm b \pm a.$$

For multiplication and division:

$$\diagup\!\!\diagdown a \diagup\!\!\diagdown b = \diagup\!\!\diagdown b \diagup\!\!\diagdown a.$$

34. Law of Distribution:

(i). For multiplication:

$$(\pm a \pm b) \times (\pm c \pm d) = +(\pm a) \times (\pm c) + (\pm a) \times (\pm d) + (\pm b) \times (\pm c) + (\pm b) \times (\pm d).$$

(ii). For division:

$$(\pm a \pm b) / (\pm c) = +(\pm a) / (\pm c) + (\pm b) / (\pm c).$$

A divisor consisting of two or more terms cannot be distributed over the dividend.

35. Laws of Exponents:

(i). Involution: $(a^{\diagup\!\!\diagdown m})^{\diagup\!\!\diagdown n} = (a^{\diagup\!\!\diagdown n})^{\diagup\!\!\diagdown m}$.

(ii). Index law: $\diagup\!\!\diagdown a^m \diagup\!\!\diagdown a^n = a^{\pm m \pm n}$.

Also, as a consequence of these two:

(iii). Corollary: $(\diagup\!\!\diagdown a \diagup\!\!\diagdown b)^m = \diagup\!\!\diagdown a^m \diagup\!\!\diagdown b^m$.

36. Laws of Logarithmic Operation:

(i). Metathesis:

$$n \log_m x = m \log_n x = \log_m x^n = \ln x^{mn}.$$

(ii). Addition theorem:

$$\log_m (x \diagup\!\!\diagdown y) = \log_m x \pm \log_m y.$$

37. Properties of 0, 1, and ∞:

$$
\begin{array}{r|r}
0 = +a-a, & 1 = \times a/a, \\
\pm b + 0 = \pm b - 0, & \diagup b \times 1 = \diagup b/1, \\
+0 = -0, & \times 1 = /1, \\
\pm\infty \pm b = \pm\infty, & 0 \diagup b = 0, \\
b - \infty = -\infty, & b/0 = b \times \infty = \times\infty, \\
b - 0 = +b. & b/\infty = \times 0.
\end{array}
$$

Zero may be regarded as the origin of additions, unity as the origin of multiplications.

38. Agenda: Involution and Logarithmic Operation in Arithmetic.

(1). Show that, if n be an integer, the index law (Art. 35) leads to the result:

$$a^n = a \times a \times a \times \ldots \text{ to } n \text{ factors},$$

and that therefore $3^2 = 9$, $2^3 = 8$, $5^3 = 125$, etc.

(2). Show, by the law of involution and the index law (Art. 35), that $8^{1/3} = 2$, $81^{1/4} = 3$, etc.

(3). Show, by the law of metathesis (Art. 36), that $^2\log 32 = 5$, $^3\log 729 = 6$, etc.

(4). Show, by the law of metathesis and the addition theorem (Art. 36), that

$$^4\log 8 + {}^4\log 2 = 2, \quad {}^4\log 2 = \tfrac{1}{2},$$
$$^6\log(1/8) + {}^6\log(1/27) = -3.$$

(5). Find the logarithms: of 16 to base $2^{1/2}$, of 125 to base $5 \times 5^{1/2}$, of 128 to modulus $1/\ln 8$, of $1/81$ to modulus $1/\ln 27$.

CHAPTER II.

GONIOMETRIC AND HYPERBOLIC RATIOS.

IX. GONIOMETRIC RATIOS.

39. Definition of Arc-Ratio. In the accompanying figures $N'N$ is a straight line fixed in position and direction, OP is supposed to have reached its position by turning about the fixed point O in the positive sense of rotation from the initial position ON. Any point

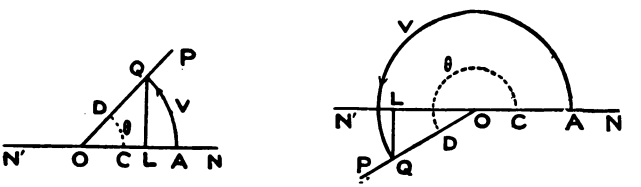

Fig. 21.

on OP at a constant distance from O describes an arc AVQ, a linear magnitude. Let the ratio of this arc to the radius OQ, both taken positively, be denoted by θ, that is,

$$\theta \equiv (\text{length of arc } AVQ) \,/\, (\text{line-segment } OQ).$$

The amount of turning of OQ, that is, the angle AOQ, fixes the value of this ratio; and since the arcs of concentric circles intercepted by common radii are proportional to those radii (Prop. 18), the ratio may be replaced by an arc CD provided only OC be taken equal to the linear unit. In

the geometrical figures a description of the angle will be sufficient to identify the ratio itself. This magnitude θ will be called the *arc-ratio* of the angle AOQ. The letter π stands for ratio of a semi-circumference to its radius, that is, the arc-ratio of 180°.

Lines drawn parallel or perpendicular to $N'N$, shall be regarded as positive when laid off from O to the right or upwards, negative when extending to the left or downwards. OP drawn outwards from O is to be considered positive in all cases.

40. Definitions of the Goniometric Ratios. LQ in the above figures being drawn perpendicular to $N'N$, upwards or downwards according as Q is above or below $N'N$ and correspondingly positive or negative, the goniometric ratios, called sine, cosine, tangent, cotangent, secant, cosecant, are defined as functions of the arc-ratio θ by the following identities:

$$\sin\theta \equiv LQ \,/\, OQ, \quad \cos\theta \equiv OL \,/\, OQ,$$
$$\tan\theta \equiv LQ \,/\, OL, \quad \cot\theta \equiv OL \,/\, LQ,$$
$$\sec\theta \equiv OQ \,/\, OL, \quad \csc\theta \equiv OQ \,/\, LQ.$$

It must be borne in mind that θ is here not an angle expressed in degrees, but a ratio, which can therefore be represented by a linear magnitude. In elementary trigonometry $\sin\theta$ usually means "sine of angle AOQ in degrees"; here it may be read "sine of magnitude θ," where $\theta = \text{arc } AVQ \,/\, \text{radius } OA$.* If v be the number of degrees in the angle AOQ, the relation between v and θ is

$$\pi v = 180\,\theta. \qquad \text{(Prop. 17.)}$$

* See Lock's *Elementary Trigonometry*, p. 31.

41. Agenda. Properties of Goniometric Ratios. Prove the following:

(1). $\sin 0 = 0, \quad \cos 0 = 1.$

(2). $\sin \frac{\pi}{2} = 1, \quad \cos \frac{\pi}{2} = 0.$

(3). $\sin^2 \theta + \cos^2 \theta = 1.$

(4). $\sec^2 \theta - \tan^2 \theta = 1.$

(5). $\csc^2 \theta - \cot^2 \theta = 1.$

If n be an integer, prove:

(6). $\sin (\theta \pm [2n + \frac{1}{2}]\pi) = \pm \cos \theta.$

(7). $\cos (\theta \pm [2n + \frac{1}{2}]\pi) = \mp \sin \theta.$

(8). $\sin (\theta \pm [2n + \frac{3}{2}]\pi) = \mp \cos \theta.$

(9). $\cos (\theta \pm [2n + \frac{3}{2}]\pi) = \pm \sin \theta.$

(10). sin, or cos of $(\theta \pm 2n\pi) =$ sin, or cos of θ.

(11). sin, or cos of $(\theta \pm [2n+1]\pi) = -$ (sin, or cos of θ).

42. Line-Representatives of Goniometric Ratios. If in the foregoing definitions the denominators OL, LQ be replaced by the radius OQ, the numerators of the six goniometric ratios will be six straight lines drawn, either from the centre, or from Q, or from one of the fixed points A, B on the circumference a quadrant's distance apart.

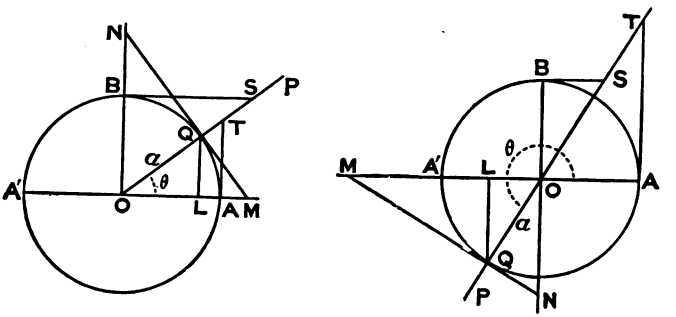

Fig. 22.

If a be the radius of the circle, they may be indicated as follows:

$a \sin \theta = LQ$, perpendicular distance of Q from $A'A$.
$a \cos \theta = OL$, distance from centre to foot of LQ.
$a \tan \theta = AT$, distance along a tangent from A to OP.
$a \cot \theta = BS$, distance along a tangent from B to OP.
$a \sec \theta = OM$, intercept of tangent at Q upon OA.
$a \csc \theta = ON$, intercept of tangent at Q upon OB.

These constructions are evidently only variations in the statement of the definitions of the goniometric ratios. When $a = 1$, the six ratios have as their geometric representatives these lines themselves.

Formerly they were defined as such for all values of the radius and were therefore not ratios, but straight lines dependent for their lengths upon the arc AQ, that is upon both the angle AOQ and the radius of the circle. The older form of definition is now rare.*

43. To Prove Limit $[(\sin \theta)/\theta] = 1$, when $\theta \doteq 0$. Let $\theta \equiv$ the arc-ratio of the angle POQ in Fig. 23, draw PQP', an arc with radius OP, draw PT and $P'T$ tangents to the arc at P and P', join P, P', and O, T. Then assuming that an arc is greater than the subtending chord and less than the enveloping tangents at its extremities, we have

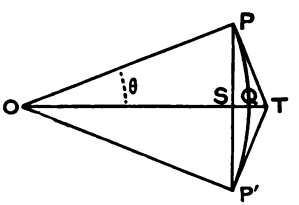

Fig. 23.

$$2\,\overline{SP} < 2\,\widehat{QP} < 2\,\overline{PT}, \quad \text{or} \quad \frac{\overline{SP}}{\overline{SP}} < \frac{\widehat{QP}}{\overline{SP}} < \frac{\overline{PT}}{\overline{SP}},$$

that is,

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

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* See Todhunter's *Plane Trigonometry*, p. 49, and the reference there given: Peacock's *Algebra*, Vol. II, p. 157. See also Buckingham's *Differential and Integral Calculus*, 3d ed., p. 139, where the older definitions are still retained.

When $\theta = 0$, $\cos\theta = 1$; therefore by making θ smaller than any previously assigned arbitrarily small magnitude, $(\sin\theta)/\theta$ is made to differ from unity by a like arbitrarily small magnitude. Under these conditions $(\sin\theta)/\theta$ is said to have 1 as its limit, and the fact is expressed by the formula

$$\lim_{\theta \doteq 0} \left\{ \frac{\sin\theta}{\theta} \right\} = 1,$$

in which $\doteq$ stands for 'approaches.'

44. Area of Circular Sector. Let the sector OAQ be divided into n equal smaller sectors by radii to the points P_1, P_2, P_3, etc., which set off the arc AQ into the same number of equal parts AP_1, P_1P_2, P_2P_3, etc., and draw P_1M perpendicular to OA. The area of each of the triangles OAP_1, OP_1P_2, OP_2P_3, . . . is $\frac{1}{2}a \cdot MP_1$,

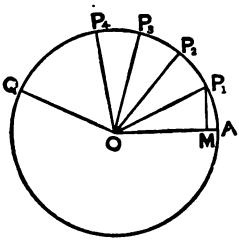

Fig. 24.

or if $(\text{arc}\,AQ)/OA = \theta$, it is $\frac{1}{2}a \cdot a\sin(AP_1/OA) = \frac{1}{2}a^2\sin(\theta/n)$, and hence the area of the entire polygon $OAP_1P_2 \ldots Q$ is

$$\frac{na^2}{2} \cdot \sin\frac{\theta}{n} = \frac{a^2\theta}{2} \cdot \frac{\sin(\theta/n)}{\theta/n}.$$

Now when the number of points of division $P_1, P_2, P_3, \ldots$ is indefinitely increased, the polygon $OAP_1P_2 \ldots Q$ approaches coincidence with the circular sector $OAP_1P_2 \ldots Q$, that is,

$$\frac{a^2\theta}{2} \cdot \frac{\sin(\theta/n)}{\theta/n} \doteq \text{area of sector } OAP_1P_2 \ldots Q, \text{ when } n \doteq \infty;$$

but at the same time

$$\theta/n \doteq 0 \text{ and } \frac{\sin(\theta/n)}{\theta/n} \doteq 1,$$

and therefore also

$$\frac{a^2\theta}{2} \cdot \frac{\sin(\theta/n)}{\theta/n} \doteq \frac{a^2\theta}{2}, \text{ when } n \doteq \infty.$$

Thus $\frac{a^2\theta}{2} \cdot \frac{\sin(\theta/n)}{\theta/n}$ can be made to differ, both from $\frac{a^2\theta}{2}$ and from the area of the circular sector, by quantities that are less than any previously assigned arbitrarily small magnitude. Under these circumstances it is assumed as axiomatic that the two limits which the varying quantity approaches cannot differ, and that therefore

$$\text{area of sector } OAP_1P_2 \ldots Q = \frac{a^2\theta}{2}.$$

The limits in fact could not be different unless the area of the sector were susceptible of two distinct values, which is manifestly impossible.

45. Agenda. The Addition Theorem for Goniometric Ratios.

From the foregoing definitions of the goniometric ratios prove for all real arc-ratios the following formulæ:

(1). $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta.$

(2). $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$

(3). $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$

(4). $\sin 2\alpha \doteq 2\sin\alpha\cos\alpha.$

(5). $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$

(6). $1 + \cos 2\alpha = 2\cos^2\alpha.$

(7). $1 - \cos 2\alpha = 2\sin^2\alpha.$

X. HYPERBOLIC RATIOS.

46. Definitions of the Hyperbolic Ratios. An important class of exponentials, which because of their relation to the equilateral hyperbola are called the hyperbolic sine, cosine, tangent, cotangent, secant and cosecant, and are symbolized by the abbreviations sinh, cosh, tanh, coth, sech, csch, are defined by the following identities:

$$\sinh u \equiv \tfrac{1}{2}(e^u - e^{-u}),$$
$$\cosh u \equiv \tfrac{1}{2}(e^u + e^{-u}),$$
$$\tanh u \equiv (e^u - e^{-u}) / (e^u + e^{-u}),$$
$$\coth u \equiv (e^u + e^{-u}) / (e^u - e^{-u}),$$
$$\operatorname{sech} u \equiv 2 / (e^u + e^{-u}),$$
$$\operatorname{csch} u \equiv 2 / (e^u - e^{-u}).$$

47. Agenda. Properties of Hyperbolic Ratios. Prove the following:

(1). $\sinh 0 = 0, \quad \cosh 0 = 1.$
(2). $\cosh^2 u - \sinh^2 u = 1.$
(3). $\operatorname{sech}^2 u + \tanh^2 u = 1.$
(4). $\coth^2 u - \operatorname{csch}^2 u = 1.$
(5). $\sinh 2u = 2 \sinh u \cosh u.$
(6). $\cosh 2u = \cosh^2 u + \sinh^2 u.$
(7). $\cosh 2u + 1 = 2 \cosh^2 u.$
(8). $\cosh 2u - 1 = 2 \sinh^2 u.$
(9). $\sinh(-u) = -\sinh u.$
(10). $\cosh(-u) = \cosh u.$

48. Geometrical Construction for Hyperbolic Ratios. For the representation of the hyperbolic ratios the equilateral hyperbola is employed. Its equation in Cartesian co-ordinates is

$$x^2 - y^2 = a^2.$$

Let OX, OY be its axes, OJ an asymptote, P any point on the curve, x and y its co-ordinates, AQB the quadrant of

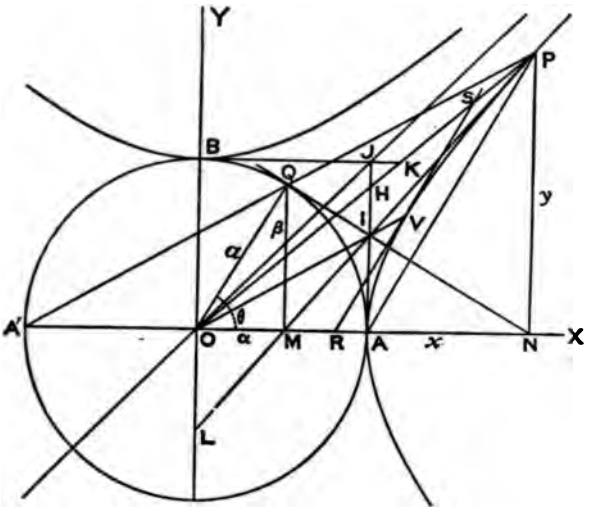

Fig. 25.

a circle with centre at the origin and radius a, NQ a tangent to the circle from the foot of the ordinate y, RS a tangent to the hyperbola parallel to the chord PA, α, β the co-ordinates of Q, θ the arc-ratio of the angle AOQ.

It is obvious from its definition that $\cosh u$ has 1 for its smallest and $+\infty$ for its largest value corresponding to $u = 0$ and ∞ respectively, and if the variations of x/a be confined to the right hand branch of the hyperbola its range of values is likewise between 1 and $+\infty$; hence we may assume

$$\frac{x}{a} = \cosh u,$$

and by virtue of the relations $\cosh^2 u - \sinh^2 u = 1$, and $x^2 / a^2 - y^2 / a^2 = 1$,

$$\frac{y}{a} = \sinh u.$$

Also, since $x^2 - y^2 = a^2$ and $x^2 - NQ^2 = a^2$, therefore

$$NQ = NP,$$

and we have

$$\frac{x}{a} = \sec\theta = \cosh u, \quad \frac{y}{a} = \tan\theta = \sinh u,$$

and the co-ordinates of Q being α, β, also

$$\frac{AH}{a} = \frac{\beta}{a} = \frac{y}{x} = \sin\theta = \tanh u, \quad \frac{\alpha}{a} = \frac{a}{x} = \cos\theta = \operatorname{sech} u,$$

and finally, since OBK is similar to ONP and $a^2 = y \cdot LO$,*

$$\frac{BK}{a} = \frac{x}{y} = \coth u, \quad \frac{LO}{a} = \frac{a}{y} = \operatorname{csch} u.$$

Hence if a be made the denominator in each of the hyperbolic ratios, their numerators will be six straight lines, drawn from O, A, B, or P, which may be indicated thus:

$a \sinh u = NP$, perpendicular distance of P from OX,
$a \cosh u = ON$, distance from centre to foot of NP,
$a \tanh u = AH$, distance along a tangent from A to OP,
$a \coth u = BK$, distance along a tangent to the conjugate hyperbola from B to OP,
$a \operatorname{sech} u = OM$, intercept of tangent at P upon OX,
$a \operatorname{csch} u = LO$, — (intercept of tangent at P upon OY).

This construction gives pertinence to the name ratio as

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*Obtained by writing the equation of the tangent MP and finding its intercept on OY; or thus, $OL / OM = NP / MN$, that is, $OL = (OM \cdot y) / (x - OM) = a (\operatorname{sech} u \sinh u) / (\cosh u - \operatorname{sech} u) = a \sinh u / (\cosh^2 u - 1) = a \operatorname{csch} u$.

applied to the six analogues of the goniometric ratios. Compare these with the constructions of Art. 42.

49. Agenda. Properties of the Equilateral Hyperbola. Prove the following propositions concerning the equilateral hyperbola. (Fig. 25 of Art. 48.)

(1). The tangent to the hyperbola at P passes through M, the foot of the ordinate to Q.

(2). The locus of I, the intersection of the tangents NQ and MP, is AJ the common tangent to the hyperbola and circle.

(3). The line OIV bisects the angle θ and the area $OAVP$ and intersects the hyperbola at its point of tangency with RS.

(4). A straight line through P and Q passes through the left vertex of the hyperbola and is parallel to OV.

(5). The angle APN = one-half the angle QON.

50. The Gudermannian. When θ is defined as a function of u by the relation $\tan\theta \equiv \sinh u$ (Art. 48) it is called the *Gudermannian* of u* and is written gd u. Sin θ, cos θ and tan θ are then regarded as functions of u and are written sg u, cg u and tg u.

51. Agenda. From the definitions of the Gudermannian functions prove the formulæ:

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* By Cayley, *Elliptic Functions*, p. 56, where the equation of definition is

$$u = \ln\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta).$$

Since $\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta) = \dfrac{1+\sin\theta}{\cos\theta}$ and $\sinh^{-1}\tan\theta = \ln\dfrac{1+\sin\theta}{\cos\theta}$ (Art. 56), the equivalence of the two definitions is obvious. The name is given in honor of Gudermann, who first studied these functions.

(1). $\mathrm{sg}^2 u + \mathrm{cg}^2 u = 1.$

(2). $\mathrm{sg}\,(u+v) = \dfrac{\mathrm{sg}\,u + \mathrm{sg}\,v}{1 + \mathrm{sg}\,u \,.\, \mathrm{sg}\,v}.$

(3). $\mathrm{cg}\,(u+v) = \dfrac{\mathrm{cg}\,u \,.\, \mathrm{cg}\,v}{1 + \mathrm{sg}\,u \,.\, \mathrm{sg}\,v}.$

(4). If $i = \sqrt{-1}$, $\quad \frac{1}{i}\,\mathrm{gd}\left(\frac{1}{i}\,\mathrm{gd}\,u\right) = -u,$

or more briefly $\qquad \left(\frac{1}{i}\,\mathrm{gd}\right)^2 u = -u.$ (Prof. Haskell.)

52. **To Prove Limit** $[(\sinh u)/u] = 1$, **when** $u \doteq 0$. In the construction of Art. 23 suppose that, during the interval of time $t' - t$, P moves over the distance $x' - x$, Q over the distance $u' - u$. Then speed being expressed as the ratio of distance passed over to time-interval, the speed of Q is

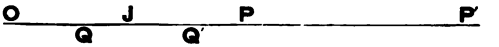

Fig. 26.

$$\mu = \frac{u' - u}{t' - t},$$

and the average speed of P during the whole interval is

$$\frac{x' - x}{t' - t} = \frac{\delta u' - \delta u}{t' - t}.$$

Let λx_0 represent the true speed of P at a given instant within the interval considered, $\lambda x_0 - \delta$, $\lambda x_0 + \delta'$ the speeds at its beginning and end respectively; then

$$\lambda x_0 - \delta < \frac{x' - x}{t' - t} < \lambda x_0 + \delta';$$

and if the interval $t' - t$ be made to decrease in such a way that δ and δ' simultaneously approach zero, the three mem-

bers of this inequality approach a common value, their limit; that is,

$$\frac{x' - x}{t' - t} = \lambda x_0, \text{ when } t' \doteq t.$$

Hence also

$$\frac{b^{u'} - b^u}{u' - u} = \frac{\lambda}{\mu} x_0, \text{ when } t' \doteq t;$$

or in the language of limits (Art. 43),

$$\underset{u' \doteq u}{\text{limit}} \left\{\frac{b^{u'} - b^u}{u' - u}\right\} = \frac{\lambda}{\mu} x_0 = \frac{x_0}{m}.$$

In the important case when $u' = -u$, and $u \doteq 0$, since $u = 0$ requires that $x = 1$, the expression last written becomes

$$\underset{u \doteq 0}{\text{limit}} \left\{\frac{b^u - b^{-u}}{2u}\right\} = \frac{1}{m},$$

and in particular when $b = e$,

$$\underset{u \doteq 0}{\text{limit}} \left\{\frac{e^u - e^{-u}}{2u}\right\} = \underset{u \doteq 0}{\text{limit}} \left\{\frac{\sinh u}{u}\right\} = 1.$$

Q. E. D. (Cf. Art. 43.)

53. **Area of a Hyperbolic Sector.** Let the perpendicular p be dropped from any point P, of the equilateral hyperbola $x^2 - y^2 = a^2$, upon its asymptote, meeting the latter in S, and let $OS = s$. The following properties of this hyperbola are well known and are proved in elementary works on conic sections:

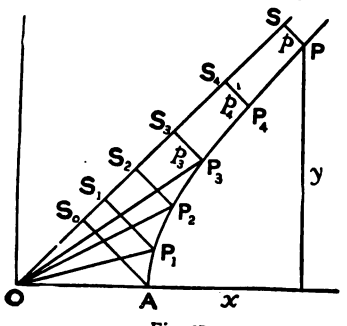

Fig. 27.

$$sp = a^2 / 2,$$
$$x + y = s\sqrt{2},$$
$$x - y = p\sqrt{2},$$
$$x = (s + p) / \sqrt{2},$$
$$y = (s - p) / \sqrt{2}.$$

Assuming these equations as known, lay off upon the asymptote

$$OS_0, OS_1, OS_2, \ldots, OS = a/\sqrt{2}, s_1, s_2, \ldots, s,$$

such that these lengths are in geometrical progression, and let the corresponding perpendiculars upon the asymptote be

$$AS_0, P_1S_1, P_2S_2, \ldots, PS = p_0, p_1, p_2, \ldots, p.$$

Then if ρ be the common ratio of the successive terms $s_0, s_1, s_2, \ldots s$, we have

$$\begin{aligned} s_0 &= a/\sqrt{2}, & p_0 &= a/\sqrt{2}, \\ s_1 &= \rho s_0, & p_1 &= a^2/(2\rho s_0), \\ s_2 &= \rho^2 s_0, & p_2 &= a^2/(2\rho^2 s_0), \\ \cdots\cdots & & \cdots\cdots\cdots & \\ s &= \rho^n s_0, & p &= a^2/(2\rho^n s_0). \end{aligned}$$

If now $(a, 0)$, (x_1, y_1), (x_2, y_2), $\ldots$, (x, y) be the co-ordinates of A, P_1, P_2, $\ldots$, P, the area of the triangle OP_2P_3 is $\frac{1}{2}(x_3y_2 - x_2y_3)$, and by virtue of the relations

$$x = (s+p)/\sqrt{2}, \quad y = (s-p)/\sqrt{2},$$

we have

$$\tfrac{1}{2}(x_3y_2 - x_2y_3) = \tfrac{1}{2}(s_2p_3 - s_3p_2);$$

but

$$s_3, \; s_2 = \rho s_2, \; \rho s_1 \text{ and } p_2, \; p_3 = p_1/\rho, \; p_2/\rho;$$

$$\begin{aligned} \therefore \quad \tfrac{1}{2}(x_3y_2 - x_2y_3) &= \tfrac{1}{2}(s_1p_2 - s_2p_1) \\ &= \tfrac{1}{2}(x_2y_1 - x_1y_2) \\ &= \text{area of triangle } OP_1P_2. \end{aligned}$$

Thus the triangles OAP_1, OP_1P_2, OP_2P_3, etc., are equal to one another in area and the area of the whole polygon $OAP_1P_2 \ldots P$ is

$$\begin{aligned} \frac{n}{2}(s_1p_0 - s_0p_1) &= \frac{n}{2}\left(s_1\frac{a}{\sqrt{2}} - p_1\frac{a}{\sqrt{2}}\right) \\ &= \frac{na^2}{4}(\rho - \rho^{-1}). \end{aligned}$$

But

$$\rho^n = \frac{s\,1/2}{a} = \frac{x}{a} + \frac{y}{a};$$

hence, assuming $x / a = \cosh u$ and $y / a = \sinh u$,

$$\rho^n = e^{nu}.$$

$$\therefore \quad \text{Area of } OAP_1 \ldots P = \frac{n a^2}{4} (e^{u/n} - e^{-u/n})$$

$$= \frac{a^2 u}{2} \cdot \frac{\sinh (u / n)}{u / n}.$$

When the number of points of division S_1, S_2, S_3, etc., is indefinitely increased the polygon $OAP_1P_2 \ldots P$ approaches coincidence with the hyperbolic sector $OAP_1P_2 \ldots P$, that is,

$$\frac{a^2 u}{2} \cdot \frac{\sinh (u / n)}{u / n} \doteq \text{area of sector } OAP_1 \ldots P, \text{ when } n \doteq \infty;$$

but at the same time

$$u / n \doteq 0 \text{ and } \frac{\sinh (u / n)}{u / n} \doteq 1,$$

and therefore

$$\frac{a^2 u}{2} \cdot \frac{\sinh (u / n)}{u / n} \doteq \frac{a^2 u}{2}, \text{ when } n \doteq \infty.$$

Hence by the reasoning of Art. 44,

$$\text{area of sector } OAP_1P_2 \ldots P = \frac{a^2 u}{2}.$$

54. Agenda. The Addition Theorem for Hyperbolic Ratios. From the foregoing definitions of the hyperbolic ratios deduce the following formulæ:

(1). $\sinh (u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v.$

(2). $\cosh (u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v.$

(3). $\tanh (u \pm v) = \dfrac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v}.$

(4). Deduce these formulæ also geometrically from the constructions of Arts. 48, 53, assuming for the definitions of sinh u and cosh u the ratios NP / a and ON / a. [Burnside: *Messenger of Mathematics*, vol. xx, pp. 145–148.]

(5). In the figure of Art. 53 show that the trapezoids $S_0AP_1S_1$, $S_1P_1P_2S_2$, etc., are equal in area to the corresponding triangles OAP_1, OP_1P_2, etc., and consequently to each other.

(6). Show that when the hyperbolic sector OAP (Art. 53) increases uniformly, the corresponding segment OS, laid off on the asymptote, increases proportionately to its own length.

(7). Assuming $a = 1$ in the equilateral hyperbola of Art. 48, and that the area of any sector is $\frac{1}{2}u$, prove that $\lim_{u \doteq 0} [(\sinh u) / u] = 1$. (Use the method of Art. 55.)

55. An Approximate Value of Natural Base. We may determine between what integers the numerical value of e must lie, by substituting their equivalents in x and y for the terms of the inequality:

Triangle $OAP >$ sector $OAVP >$ triangle ORS,

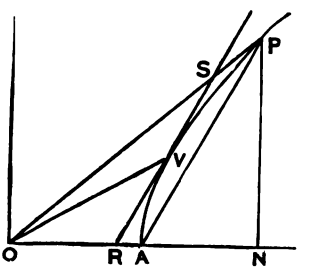

Fig. 28.

as represented in Fig. 28. For our present purpose it will involve no loss of generality and it will simplify the computation to assume $OA = 1$, so that the equation of the hyperbola is

$$x^2 - y^2 = 1.$$

The sectorial area $OAVP$, as previously found in Art. 53, is then $\frac{1}{2}u$, the area of OAP is obviously $\frac{1}{2}y$, and for that of ORS we may write

$$\tfrac{1}{2}\, OR \times (\text{ordinate of } S).$$

To determine OR and this ordinate, write the equations to the tangent RS and the line OP, and find the ordinate of their intersection, and the intercept of the former on the x-axis. The results are:

$$\eta = \frac{y}{x-1}\xi - \sqrt{2/(x-1)},$$

the equation to the tangent RS, ξ and η being the current co-ordinates of the line;

$$\eta = \frac{y}{x}\xi,$$

the equation to OP;

$$OR = \frac{\sqrt{2(x-1)}}{y},$$

the required intercept on the x-axis; and

$$\eta = \sqrt{2(x-1)},$$

the ordinate of S, found by eliminating ξ from the equations to RS and OP. Hence the area ORS is

$$\tfrac{1}{2}\eta \cdot OR = \frac{x-1}{y},$$

and the inequality between the areas takes the form

$$y > u > 2\frac{x-1}{y}.*$$

If $y = 1$, then $u < 1$, $x = \sqrt{2}$, and

$$e^u = x + y = 1 + \sqrt{2} = 2.4+;$$

therefore $$e > 2.4.$$

* Had the assumption $a = 1$ not been made, this inequality would have been

$$\frac{y}{a} > u > 2\frac{x-a}{y}.$$

The equations for RS and OP and the expressions for OR and η would have been correspondingly changed, but the final results would have been the same as those given above.

If $2\frac{x-1}{y}=1$, then $u>1$, $x=\frac{5}{3}$, $y=\frac{4}{3}$, and

$$e^u=x+y=3;$$

therefore $$e<3.$$

A nearer approximation to the value of e is found by other methods. To nine decimal places it is 2.718281828.

56. **Agenda.** Logarithmic Forms of Inverse Hyperbolic Ratios. It is customary to represent by $\sinh^{-1}y$, $\cosh^{-1}x$, . . . , the arguments whose sinh, cosh, . . . , are y, x, etc. Let $t\equiv\sinh^{-1}y$; then

$$y=\sinh t=\frac{1}{2}(e^t-e^{-t}),$$

whence, multiplying by e^t and re-arranging terms,

$$e^{2t}-2ye^t-1=0,$$

a quadratic equation in e^t, the solution of which gives

$$e^t=y\pm\sqrt{y^2+1},$$

or $$t=\ln(y\pm\sqrt{y^2+1}).$$

If y be real, the upper sign must be chosen; for $\sqrt{y^2+1}>y$ and e^t is positive for all real values of t (Art. 23). Hence

(1). $\sinh^{-1}y=\ln(y+\sqrt{y^2+1})$.

Prove by similar methods the following formulæ:

(2). $\cosh^{-1}x=\ln(x+\sqrt{x^2-1})$.

(3). $\tanh^{-1}z=\frac{1}{2}\ln\frac{1+z}{1-z}$.

(4). $\coth^{-1}z=\frac{1}{2}\ln\frac{z+1}{z-1}$.

(5). $\operatorname{sech}^{-1}x=\ln(1/x+\sqrt{1/x^2-1})$.

(6). $\operatorname{csch}^{-1}y=\ln(1/y+\sqrt{1/y^2+1})$.

(Cf. Art. 97.)

CHAPTER III.

THE ALGEBRA OF COMPLEX QUANTITIES.

XI. GEOMETRIC ADDITION AND MULTIPLICATION.

57. Classification of Magnitudes: Definitions. It was pointed out in Art. 2 that any one straight line suffices for the complete characterization of all so-called real quantities; in fact the real magnitudes of algebra were defined as lengths set off upon such a line. But because, in this representation, no distinction in direction was necessary, all line-segments were taken to be real magnitudes, and comparisons of direction were made, by means of the principles of geometrical similarity, for the sole purpose of determining lengths. Such comparisons will still be necessary whenever the product or quotient of two real magnitudes is called for, but into the real magnitudes themselves no element of direction enters; their sole characteristics are length and sense, that is, length and extension forwards or backwards.

If the attempt be made to apply the various algebraic processes to all real magnitudes, negative as well as positive, another kind of magnitude, not yet considered, is necessarily introduced. For example, if x be positive, no real magnitude can be made to take the place of either $(-x)^{1/2}$ or $\log_m(-x)$; for the square of a real quantity is always positive (Art. 14), and the definition of an exponential given in Art. 23 precludes its ever assuming a negative value. In order that forms like these may be admitted into

the category of algebraic quantity, a new kind of quantity must therefore be defined, or more properly, a new definition of algebraic quantity in general must be given.

Having assigned some fixed direction as that in which all real quantities are to be taken, we adopt a straight line having this direction as a line of reference, call it the real axis, and determine the directions of all other straight lines in the plane by the angles they make with this fixed one. Line-segments having directions other than that of the real axis are the new magnitudes that now demand consideration. They are called *vectors*. They have two determining elements: length and the angle they make with the real axis.

(i). Its length, taken positively, is called the *tensor* of the magnitude, and the arc-ratio of the angle it makes with the real axis is called its *amplitude* (or *argument*).

Classified and defined with respect to amplitude, the magnitudes themselves are:

(ii). *Real*, if the amplitude be 0 or a multiple of π;

(iii). *Imaginary*, if the amplitude be $\pi/2$ or an odd multiple of $\pi/2$.

(iv). *Complex*, for all other values of the amplitude.

In general, therefore, vectors in the plane represent complex quantities, but in particular, when parallel to the real axis they represent real quantities; when perpendicular to it, imaginary.

Any quantity is by definition uniquely determined by its tensor and amplitude, and hence:

(v). Two quantities are equal if their tensors and their amplitudes are respectively equal, the geometrical rendering of which is: two magnitudes, or vectors, are equal if (and only if) they are at once parallel, of the same sense, and of equal lengths.

The algebra of complex quantities, like that of real quantities, is developed from the definitions of the fundamental algebraic operations: addition and subtraction, multiplication and division, exponentiation and the taking of logarithms. These operations applied to magnitudes represented by straight lines in the plane are called algebraic by reason of their identity with those of the analysis of real quantities, but specifically geometric, because each individual operation has its own unique geometrical configuration. On the other hand, the algebraic processes applied to real quantities may be described as geometric addition, multiplication, involution, etc., in a straight line.

58. Geometric Addition. Regarding lines for our present purpose as generated by a moving point, the operation of addition is defined to mean that a point P, free to move in any direction, is successively transferred forwards or backwards, that is, in the positive or negative sense as marked by the signs $+$ and $-$, through certain distances designated by appropriate symbols α, β, γ . . Thus the sum $+\alpha - \beta + \gamma$, in which α, β, γ represent vectors in the plane (or in space), joined to form a zig-zag, as shown in the accompanying figure, may be read off as follows, the arrow-heads indicating direction of motion forwards: Move forwards through distance α to A, then backwards through distance β to B', then forwards through distance γ to C', and the result is the same as if the motion had taken place in a direct line from O to C'; this fact is expressed in the equation

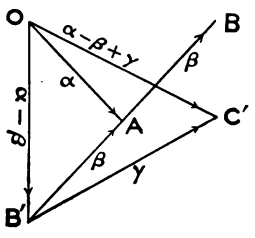

Fig. 29.

$$+\alpha - \beta + \gamma = \overline{OC'}.$$

If not already contiguous, the magnitudes that form the terms of a sum, by changing the positions of such as require it without changing their direction, may be so placed that all the intermediate extremities are conterminous. Geometric addition may therefore be defined as follows:

The sum of two or more magnitudes, placed for the purpose of addition so as to form a continuous zig-zag, is the single magnitude that extends from the initial to the terminal extremity of the zig-zag.

59. **The Associative and Commutative Laws** for geometric addition in the plane are deduced as immediate consequences of its definition. For in the first place, the ultimate effect is the same whether a transference is made from O direct to B then to C as expressed by $(\alpha+\beta)+\gamma$, in the subjoined figure, or from O to A then direct from A to C as expressed by $\alpha+(\beta+\gamma)$, or from O to A to B to C as expressed by $\alpha+\beta+\gamma$; hence

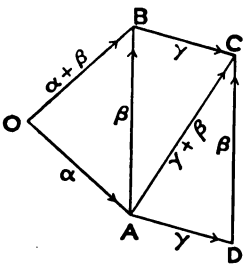

Fig. 30.

$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$$
$$=\alpha+\beta+\gamma;$$

and in the second place, by (v) of Art. 57, in $ABCD$, a parallelogram, $\overline{AD}=\overline{BC}=\gamma$, $\overline{DC}=\overline{AB}=\beta$, and by the definition of addition $\overline{AB}+\overline{BC}=\overline{AC}=\overline{AD}+\overline{DC}$, whence

$$\beta+\gamma=\gamma+\beta.$$

One or more of the terms may be negative. Expressing this fact by writing $\pm\alpha$, $\pm\beta$, $\pm\gamma$ in place of α, β, γ, the two resultant equations of the last paragraph become

$$(\pm\alpha\pm\beta)\pm\gamma=\pm\alpha+(\pm\beta\pm\gamma)=\pm\alpha\pm\beta\pm\gamma,$$
$$\pm\gamma\pm\beta=\pm\beta\pm\gamma,$$

which express the associative and commutative laws for addition and subtraction.

It is evident that the principles of geometric addition apply equally to vectors in space, or in the plane, or to segments of one straight line. In particular algebraic addition (Art. 3) may be described as geometric addition in a straight line.

60. Geometric Multiplication. The geometric product of two magnitudes α, β, is defined as a third magnitude γ, whose tensor is the algebraic product of the tensors of the factors and whose amplitude is the algebraic sum of their amplitudes, constructed by the rules for algebraic product and sum. (Arts. 5, 3.)

If one of the factors be real and positive, the amplitude of the other reappears unchanged as the amplitude of the product, which is then constructed, by the algebraic rule, upon the straight line that represents the direction of the complex factor, and it was proved in Art. 18 that in such a construction an interchange of factors does not change the result. Hence, if a be real and positive and β complex,

$$a \times \beta = \beta \times a.$$

In this product, tensor of $a \times \beta = a \times$ tensor of β by definition, and if tensor of $\beta = 1$, the complex quantity $a \times \beta$ appears as the product of its tensor and a unit factor β, a complex unit, which when applied as a multiplier to a real quantity a, does not change its length but turns it out of the real axis into the direction of β. Any such complex unit is called a *versor*. Let tsr stand for tensor, vsr for versor; then every complex quantity α can be expressed in the form

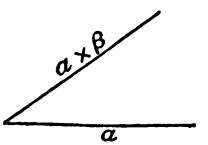

Fig. 31.

$$\alpha = \text{tsr } \alpha \times \text{vsr } \alpha.$$

This versor factor is wholly determined by its amplitude, in terms of which it is frequently useful to express it. For this purpose let i be the versor whose amplitude is $\pi/2$, θ the amplitude of the complex unit β, OX the real axis, BM the perpendicular to OX from the terminal extremity of β. Then $MB = \sin\theta$, $OM = \cos\theta$, and by the rule of geometric addition.

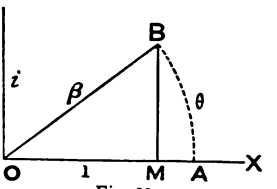

Fig. 32.

$$\beta = OM + i \times MB = \cos\theta + i\sin\theta.$$

As an abbreviation for $\cos\theta + i\sin\theta$ it is convenient to use $\operatorname{cis}\theta$, which may be read: *sector of* θ. In this symbolism, the law of geometric multiplication (product of complex quantities, as above defined) is expressed in the formula,

$$(a \cdot \operatorname{cis}\phi) \times (b \cdot \operatorname{cis}\psi) = a \times b \cdot \operatorname{cis}(\phi + \psi).$$

It is obvious that algebraic multiplication, described in Art. 5, is a particular form of geometric multiplication, being geometric multiplication in a straight line.

61. Conjugate and Reciprocal. If in the last equation $b = a$ and $\psi = -\phi$, it becomes

$$(a \cdot \operatorname{cis}\phi) \times (a \cdot \operatorname{cis}[-\phi]) = a \times a \cdot \operatorname{cis} 0 = a^2.$$

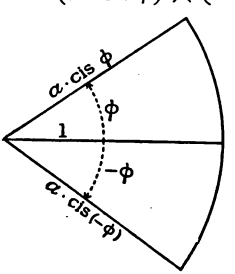

Fig. 33.

The factors of this product, $a\operatorname{cis}\phi$ and $a\operatorname{cis}(-\phi)$, are said to be *conjugate* to one another, and we have the rule:

The product of two conjugate complex quantities is equal to the square of their tensor.

If this tensor be 1, the product reduces to

$$\operatorname{cis}\phi \cdot \operatorname{cis}(-\phi) = 1;$$

that is, by virtue of the definition of Art. 6, cis ϕ and cis $(-\phi)$ are reciprocal to each other and we may write

$$/ \operatorname{cis} \phi = \operatorname{cis} (-\phi).$$

In like manner, since it is now evident that

$$(a \cdot \operatorname{cis} \phi) \times (/ a \cdot / \operatorname{cis} \phi) = a / a \cdot \operatorname{cis} 0 = 1$$
$$\therefore \quad / (a \cdot \operatorname{cis} \phi) = / a \cdot / \operatorname{cis} \phi.$$

If now in the formula expressing the law of multiplication we write $/ b$ and $-\psi$ for b and ψ respectively, we have, as the law of geometric division,

$$(a \cdot \operatorname{cis} \phi) / (b \cdot \operatorname{cis} \psi) = a / b \cdot \operatorname{cis} (\phi - \psi).$$

62. Agenda. Properties of cis ϕ. If n be an integer prove:

(1). $(a \cdot \operatorname{cis} \phi)^n = a^n \cdot \operatorname{cis} n\phi$.

(2). $\operatorname{cis} (\phi \pm 2n\pi) = \operatorname{cis} \phi$.

(3). $\operatorname{cis} (\phi \pm [2n + 1]\pi) = -\operatorname{cis} \phi$.

(4). $\operatorname{cis} (\phi + [2n \pm \tfrac{1}{2}]\pi) = \pm i \operatorname{cis} \phi$.

(5). $\operatorname{cis} (\phi + [2n \pm \tfrac{3}{2}]\pi) = \mp i \operatorname{cis} \phi$.

(6). Show that the ratio of two complex quantities having the same amplitude, or amplitudes that differ by $\pm 2\pi$, is a real quantity.

(7). Show that the ratio of two complex quantities having amplitudes that differ by $\pm \tfrac{1}{2}\pi$ is a purely imaginary quantity.

63. The Imaginary Unit. By definition (iii) of Art. 57 cis $\pi/2 = i$ is an imaginary having a unit tensor; it is therefore called the imaginary unit. Its integral powers form a closed cycle of values; thus:

$$i = \operatorname{cis} \frac{\pi}{2},\ i^2 = \operatorname{cis} \pi = -1,\ i^3 = \operatorname{cis} \frac{3\pi}{2} = -i,\ i^4 = \operatorname{cis} 2\pi = 1,$$

and the higher powers of i repeat these values in succession; that is, if n be an integer,

$$i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i, \quad i^{4n+4} = 1,$$

and these are the only values the integral powers of i can acquire.

64. The Associative and Commutative Laws. Let a, b, c be the tensors, ϕ, ψ, χ the amplitudes of α, β, γ respectively; that is,

$$\alpha = a \cdot \operatorname{cis}\phi, \quad \beta = b \cdot \operatorname{cis}\psi, \quad \gamma = c \cdot \operatorname{cis}\chi.$$

Then, by the law of geometric multiplication,

$$\begin{aligned} \alpha \times (\beta \times \gamma) &= (a \cdot \operatorname{cis}\phi) \times ([b \cdot \operatorname{cis}\psi] \times [c \cdot \operatorname{cis}\chi]) \\ &= (a \cdot \operatorname{cis}\phi) \times (b \times c \cdot \operatorname{cis}[\psi + \chi]) \\ &= a \times (b \times c) \cdot \operatorname{cis}(\phi + [\psi + \chi]); \end{aligned}$$

and by the same process,

$$(\alpha \times \beta) \times \gamma = (a \times b) \times c \cdot \operatorname{cis}([\phi + \psi] + \chi).$$

But, by the rules of algebraic multiplication and addition,

$$a \times (b \times c) = (a \times b) \times c \text{ and } \phi + (\psi + \chi) = (\phi + \psi) + \chi;$$
$$\therefore \quad \alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma,$$

which is the associative law for multiplication.

And again, by the law of geometric multiplication,

$$\begin{aligned} \alpha \times \beta &= (a \cdot \operatorname{cis}\phi) \times (b \cdot \operatorname{cis}\psi) \\ &= a \times b \cdot \operatorname{cis}(\phi + \psi), \end{aligned}$$

and similarly $\beta \times \alpha = b \times a \cdot \operatorname{cis}(\psi \times \phi)$.

But, by the rules of algebraic multiplication and addition,

$$a \times b = b \times a \text{ and } \phi + \psi = \psi + \phi;$$
$$\therefore \quad \alpha \times \beta = \beta \times \alpha,$$

which is the commutative law for multiplication.

The letters here involved may obviously represent either direct factors or reciprocals, and the sign $\times$ may be replaced at pleasure by the sign /, without affecting the proof here given (see Art. 61). Hence, as in Arts. 32, 33, with real magnitudes, so with complex quantities the associative and commutative laws for multiplication and division have their full expression in the formulæ,

$$\because(\because\alpha\because\beta)=\because(\because\alpha)\because(\because\beta),$$
$$\because\alpha\because\beta=\because\beta\because\alpha.$$

65. The Distributive Law. From the definitions of geometric addition and multiplication (Arts. 58, 60) the law of distribution for complex quantities is an easy consequence. The constructions of the subjoined figure, in which α, β and $m\cdot\operatorname{cis}\theta$ represent complex quantities will bring this law in direct evidence. The operation $m\cdot\operatorname{cis}\theta$ changes $\overline{OA}$ into $\overline{OA'}$, $\overline{AB}=\overline{A'F}$ into $\overline{AE}=\overline{A'B'}$, and $\overline{OB}$ into $\overline{OB'}$, that is, turns each side of the triangle OAB through the angle AOA' and changes its length in the ratio of m to 1, producing the similar triangle $OA'B'$, in which

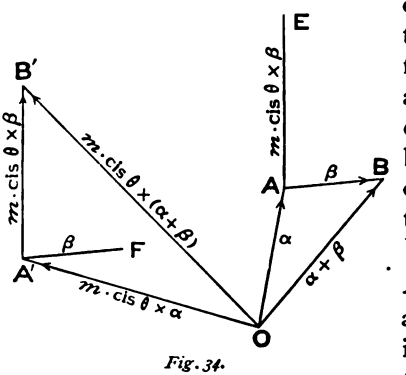

Fig. 34.

$$\overline{OA'}=m\cdot\operatorname{cis}\theta\times\alpha,$$
$$\overline{A'B'}=m\cdot\operatorname{cis}\theta\times\beta,$$
$$\overline{OB'}=m\cdot\operatorname{cis}\theta\times(\alpha+\beta).$$

But, by the rule of geometric addition,

$$\overline{OB'} = \overline{OA'} + \overline{A'B'};$$

$$\therefore \quad m \cdot \operatorname{cis}\theta \times (\alpha + \beta) = m \cdot \operatorname{cis}\theta \times \alpha + m \cdot \operatorname{cis}\theta \times \beta.$$

This demonstration is in no way disturbed by the introduction of negative and reciprocal signs. The last equation above written is, in fact, the first equation of page 37, and the subsequent equations of Arts. 20, 21 and their proofs remain intact when for the real quantities a, b, c, d, etc., complex quantities are substituted. Hence, writing $m \cdot \operatorname{cis}\theta = \gamma$,

$$(\pm\alpha \pm \beta) \diagup\!\!\diagdown (\pm\gamma) = + (\pm\alpha) \diagup\!\!\diagdown (\pm\gamma) + (\pm\beta) \diagup\!\!\diagdown (\pm\gamma).$$

Here, as in Arts. 20, 21, the sign $\times$ is distributive over two or more terms that follow it, but not so the sign /.

66. Argand's Diagram. It is obvious from its definition as here given (Art. 57) that to every complex quantity there corresponds in the plane a unique geometrical figure which completely characterizes it. This figure is known as Argand's diagram,* and consists of the real axis OX with reference to which the arc-ratio θ is estimated, the imaginary axis OY perpendicular to OX, the directed line OP that represents the complex quantity and the perpendicular PA from P to OX. The axes OX and OY are supposed to be fixed in position and direction for all quantities. Any point P in the plane then determines one and only one complex quantity and one set of line-segments OA, AP, OP, different from every other set.

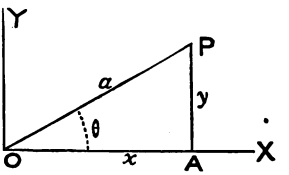

Fig. 35.

* First constructed for this purpose by Argand: *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques;* Paris, 1806. Translated by Prof. A. S. Hardy, New York, 1881.

If $OA = x$ and $AP = y$, the complex quantity appears in the form $x + iy$, and if $OP = a$ and arc-ratio of $AOP = \theta$, the relations

$$x = a\cos\theta,\ y = a\sin\theta,$$
$$a^2 = x^2 + y^2,\ \tan\theta = y/x$$

are directly evident from the figure and we have

$$x + iy = a(\cos\theta + i\sin\theta),$$
$$\text{tsr}(x + iy) = +\sqrt{x^2 + y^2},$$
$$\text{vsr}(x + iy) = \cos\theta + i\sin\theta,$$
$$\text{amp}(x + iy) = \text{arc-ratio whose tangent is } y/x.$$

In analysis the complex quantity most frequently presents itself either explicitly in the form $x + iy$, or implicitly in some operation out of which this form issues.

67. Agenda. Multiplication, Division and Construction of Complex Quantities. Prove the following:

(1). $(a + ib) \times (x + iy) = ax - by + i(ay + bx)$.

(2). $(a+ib)/(x + iy) = \dfrac{ax + by + i(bx - ay)}{x^2 + y^2}$.

(3). $(a + ib)^4 + (a - ib)^4 = 2(a^4 + b^4) - 12a^2b^2$.

(4). $\left(\dfrac{a + ib}{a - ib}\right)^2 - \left(\dfrac{a - ib}{a + ib}\right)^2 = \dfrac{4iab}{(a^2 + b^2)^2}$.

(5). $\dfrac{1 + 2i}{1 + i} + \dfrac{1 - 2i}{1 - i} = 3$.

(6). $\dfrac{1}{(x + iy)^2} = \dfrac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2}$.

(7). $\dfrac{x + iy}{(x + y/i)^2} = \dfrac{x^3 - 3xy^2 + i(3x^2y - y^3)}{(x^2 + y^2)^2}$.

(8). $(-\frac{1}{2} + i\frac{1}{2}\sqrt{3})^2 = -\frac{1}{2} - i\frac{1}{2}\sqrt{3}$.

(9). $(-\frac{1}{2}+i\frac{1}{2}\sqrt{3})^3=(-\frac{1}{2}-i\frac{1}{2}\sqrt{3})^3=1.$

(10). $\sqrt{x+iy}$
$=\pm\frac{1}{2}\left\{\sqrt{\sqrt{(x^2+y^2)}+x}+i\sqrt{\sqrt{(x^2+y^2)}-x}\right\}.$

(11). $[\pm(1+i)/\sqrt{2}]^4=[\pm(1-i)/\sqrt{2}]^4=-1.$

(12). Write down the expression for tensor in each of the above examples.

(13). Prove, by the aid of Argand's diagram (Art. 66), that the tensor of the sum of two or more complex quantities cannot be greater than the sum of their tensors; that is,

$$\text{tsr}(\alpha+\beta)\leqq\text{tsr}\,\alpha+\text{tsr}\,\beta.$$

(14). By definition (Art. 60),

$$\text{tsr}(\alpha\diagup\!\!\diagdown\beta)=\text{tsr}\,\alpha\diagup\!\!\diagdown\text{tsr}\,\beta,$$
and
$$\text{amp}(\alpha\diagup\!\!\diagdown\beta)=\text{amp}\,\alpha\pm\text{amp}\,\beta,$$

and hence no proofs of these properties are called for.

Construct the following, applying for the purpose the rules of algebraic and geometric addition and multiplication (Arts. 3, 5, 58, 60):

(15). $(a+ib)+(x+iy).$
(16). $(a+ib)-(x+iy).$
(17). $(a+ib)\times(x+iy).$
(18). $(a+ib)/(x+iy).$
(19). $(x+iy)^2.$
(20). $1/(x+iy).$
(21). $(a\cdot\text{cis}\,\phi)\times(b\cdot\text{cis}\,\psi).$
(22). $b/(a\cdot\text{cis}\,\phi).$
(23). $(a\cdot\text{cis}\,\phi)^3.$
(24). $(a\cdot\text{cis}\,\phi)/(b\cdot\text{cis}\,\psi)^2.$

XII. EXPONENTIALS AND LOGARITHMS.*

68. Definitions. In a circle whose radius is unity, OT is assumed to have a fixed direction, its angle with the real axis $= JOT$, (Fig. 36), OR is supposed to turn about

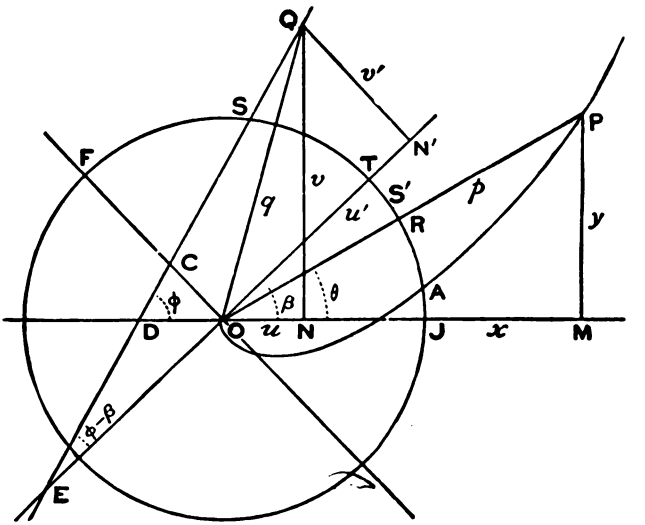

Fig. 36.

O with a constant speed, Q to move with a constant speed along any line, as ES, in the plane, P along OR with a speed proportional to its distance from O. Let

* The theory of logarithms and exponentials, as here formulated, was the subject-matter of a paper by the author, entitled "The Classification of Logarithmic Systems," read before the New York Mathematical Society in October, 1891, and subsequently published in the *American Journal of Mathematics*, Vol. XIV, pp. 187-194. It was further discussed by Professor Haskell and by the author in two notes in the *Bulletin of the New York Mathematical Society*, Vol. II, pp. 164-170.

speed of P in OR at $A \equiv \lambda$, arc-ratio of $JOR \equiv \theta$,
speed of Q in ES $\equiv \mu$, arc-ratio of $JDS \equiv \phi$,
speed of R in JRS $\equiv \omega$, arc-ratio of $JOT \equiv \beta$,

$$OP,\ OQ \equiv p,\ q,\quad OM,\ MP \equiv x, y,$$
$$ON,\ NQ \equiv u,\ v,\quad ON',\ N'Q \equiv u',\ v',$$
$$OC \equiv c,\quad JA \equiv \alpha - (\text{a possible multiple of } 2\pi),$$
$$\mu / \sqrt{\lambda^2 + \omega^2} \equiv m,$$
$$\overline{OT} \equiv \cos\beta + i \sin\beta \equiv \operatorname{cis}\beta,\quad m \operatorname{cis}\beta \equiv \kappa,$$
$$x + iy \equiv z,\quad u + iv \equiv w.$$

ET will be called the *modular line*, and OF, drawn through the origin perpendicular to ET, will be called the *modular normal.*

In all logarithmic systems the relation

$$\omega / \lambda = \tan(\phi - \beta)$$

is assumed to exist, and this, together with the equation $\mu / \sqrt{\lambda^2 + \omega^2} = m$, by elimination of ω, gives, as a second expression for m,

$$m = \mu / \lambda \cdot \cos(\phi - \beta).$$

Let the values of m and β be assigned, and the path and speed of Q determined, by fixing the angle JDS, the position of the point C and the value of μ. The value of λ is then completely determined through the equation $m = \mu / \lambda \cdot \cos(\phi - \beta)$, and the value of ω by the previous equation $\omega = \lambda \tan(\phi - \beta)$. Thus the curve upon which P moves, when Q moves upon a known straight line, is given its definite form by the values assigned to m and β, which are therefore the two independent determining factors in any logarithmic system.

But it is still unknown whether, when the position of Q is assigned, P is far or near, and in order to completely define the position of P relatively to that of Q let it be

agreed that when Q traverses some specified distance along its straight path P shall pass through a definite portion of its curved path; whence it will then follow that to every position of Q corresponds a known unique and determinate position of P.

A convenient assumption for this purpose is found to be that while Q is passing from the modular line to the modular normal (E to C), P shall go from a point on the real axis to the circumference of the unit circle, upon a segment of its path, which, if necessary, may wind about the origin one or more times; θ increasing meanwhile from o to JA or from o to JA plus a multiple of 2π. This assumption determines C and A as corresponding points and fixes A as a definite point on the circumference of the unit circle (see Art. 72).

Having thus set up a unique or one-to-one correspondence between the positions of P and Q in their respective paths, we define the terms modulus, base, exponential and logarithm as follows:

(i). The *modulus* is the product of the two independent quantities m and cis β; that is, if $\kappa \equiv$ modulus,

$$\kappa = \mu / \sqrt{\lambda^2 + \omega^2} \cdot (\cos\beta + i \sin\beta)$$
$$= \mu / \lambda \cdot \cos(\phi - \beta) \cdot (\cos\beta + i \sin\beta).$$

(ii). The *base* is the value that $\overline{OP}$, or $x + iy$, assumes at the instant when $\overline{OQ}$ becomes 1, that is, when Q passes through the point J as it moves along some line that intersects the unit circle at J. In general B will stand for base corresponding to modulus κ.

(iii). OP is the *exponential* of OQ, either with respect to the modulus κ, as expressed by the identity

$$x + iy \equiv \exp_\kappa (u + iv),$$

or with respect to the base B, as expressed by the identity

$$x + iy \equiv B^{u+iv}.$$

(iv). Inversely, OQ is the *logarithm* of OP, either with respect to the modulus κ, as expressed by the identity

$$u + iv \equiv \log_\kappa (x + iy),$$

or with respect to the base B, as expressed by the identity

$$u + iv \equiv {}^B\log (x + iy).$$

69. Exponential of 0, 1, and Logarithm of 1, B. If the path of Q pass through the origin, the points E and C will coincide at O and the path of P will cross both the circumference of the unit circle and the real axis at J. Hence

$$y = 0 \text{ and } x = 1, \text{ when } u = v = 0,$$

to which correspond the convenient relations

$$\exp_\kappa 0 = B^0 = 1,$$

and

$$\log_\kappa 1 = {}^B\log 1 = 0.$$

Here also, as in Art. 23, because $w = 1$ when $z = B$,

$$\therefore \quad \log_\kappa B = 1, \text{ and } \exp_\kappa 1 = B^1 = B.$$

70. Classification of Systems. The special value zero for the modular angle JOT' eliminates the imaginary term from the modulus and introduces the ordinary system of logarithms, with a real modulus. A system is called *gonic*, or *a-gonic*, according as its modulus does or does not involve the angular element β.

The geometrical representation of agonic systems is obtained from Fig. 36 by turning the rigidly connected group of lines EN', EQ and OF, together with the specified points upon them, around the origin as a fixed centre,

backwards through the angle JOT, so that T, S fall into the positions J, S'. D and E then coincide upon JD, OF becomes perpendicular to JD, $\phi - \beta$ remains unchanged in value but merges into ϕ, and the new modulus becomes $\mu / \sqrt{\lambda^2 + \omega^2}$, or $\mu / \lambda \cdot \cos \phi$, its former value with the factor cis β omitted; while no change in λ, μ and ω need take place. Thus the path of P remains intact, and the new Q moves with its former speed in a straight line passing through S' and through a point on OJ at a distance to the left of O equal to OE.

Hence the values of z in the two systems are identical, while w, of the original gonic system, in virtue of the backward rotation through the angle JOT, is transferred to the new or agonic system, by being multiplied by cis $(-\beta)$, so that the original w and its transformed value, here denoted by w', bear to one another the relation

$$w = w' \text{ cis } \beta.$$

The agonic system above described obviously has for its equations of definition (Art. 68)

$$w' = \log_m z, \quad z = b^{w'},$$

in which $m = \mu / \lambda \cdot \cos \phi$ and b is the value of z for which $\log_m z = 1$. The formula connecting logarithms in the two systems therefore is

$$\log_\kappa z = \text{cis } \beta \cdot \log_m z. \qquad [\kappa = m \text{ cis } \beta.]$$

Finally, if in this equation $z = B$, the resulting relation between β and B is

$$\text{cis } (-\beta) = \log_m B,$$

or in the inverse form it is

$$B = \exp_m \{\text{cis } (-\beta)\} = b^{\text{cis}(-\beta)}.$$

71. **Special Constructions.** A further specialization is obtained by making $\phi = \beta = 0$. Q then moves upon a line parallel to OJ, and since $\tan(\phi - \beta)$ is now zero, ω is also zero, R remains fixed at A, and P moves upon a straight line passing through O and A. This is a convenient construction for logarithms and exponentials of complex quantities with respect to a real modulus. The base is here also real (Art. 70).

In particular, if Q moves upon the real axis itself, A coincides with J, P also moves upon the real axis, and the resulting construction is that described in Art. 23 for logarithms of real quantities.

Returning to the general case in which β is not zero, we are at liberty, by our original hypothesis concerning the motion of Q, to permit Q to move upon any straight line in the plane, provided we assign to P such a motion as shall consist with the definition that OQ shall be the logarithm, to modulus κ, of OP. Accordingly, let Q be supposed to move parallel to EN'; $\phi - \beta$ is then zero, ω is likewise zero, and P moves upon a straight line passing through O and A. Or again, let Q move parallel to OF; $\phi - \beta$ is then $\pi/2$, λ is zero, and P moves in a circumference concentric with the unit circle. Such constructions are possible to every logarithmic system and enable us to simplify tbe graphical representation of the relative motions of P and Q.*

Expressions of the form $\log_\kappa x$, for which no interpretation could be found in terms of real quantities unless x were real and positive (Art. 57), will henceforth be susceptible of definite geometrical representation for all possible values of x. See the examples of Art. 86.

*We might, in fact, propose to assign, as the path of P in the first instance, any straight line passing through the origin, define OP as the exponential of OQ, and then determine the modulus by the appropriate auxiliary construction.

72. Relative Positions of A and C in Fig. 36. We have by definition,

$$c \equiv OC, \quad a \equiv JA + \text{a possible multiple of } 2\pi,$$
$$m \equiv \mu / \lambda \cdot \cos(\phi - \beta),$$
$$\omega = \lambda \tan(\phi - \beta).$$

The product of the last two of these equations gives

$$m\omega = \mu \sin(\phi - \beta).$$

But ω and $\mu \sin(\phi - \beta)$ are the rates of change of θ and v' respectively, and $\theta = 0$, $v' = 0$ are simultaneous values (Art. 68),

$$\therefore \quad m\theta = v';$$

and since $\theta = a$, $v' = c$ are also simultaneous values (Art. 68),

$$\therefore \quad ma = c.$$

Thus A has always a definite position depending upon the modulus and the distance from the origin at which Q crosses the modular normal.

Since a is the length of arc over which R passes while Q passes from E to C, it is evident that when c/m lies between $2k\pi$ and $2(k+1)\pi$, say

$$c/m = 2k\pi + (<2\pi)$$

where k is an integer, the part of P's path that corresponds to EC encircles the origin k times before it intersects the circumference of the unit circle, and the point upon the real axis that corresponds to E is its $(k+1)^{\text{th}}$ intersection with the path of P, counting from J to the left.

73. The Exponential Formula. If $\theta =$ arc-ratio of MOP,

$$z = p \operatorname{cis} \theta = B^{u+iv}.$$

It is required to find p and θ as functions of u and v.

Since the speed of N' in OT is $\mu \cos(\phi - \beta)$ and that of P in OR is λp, and since by definition $m = \mu / \lambda \cdot \cos(\phi - \beta)$, the relation between ON', $= u'$ and OP, $= p$, two real quantities, is that of an exponential to its logarithm, with respect to the modulus m (Art. 23); that is, if $b =$ base corresponding to modulus m,

$$u' = \log_m p, \text{ and } p = b^{u'};$$

and in Art. 72 it was shown that

$$\theta = v' / m.$$

Hence
$$B^{u+iv} = b^{u'} \operatorname{cis} \frac{v'}{m}.$$

But by considering the projections of u, v upon u', v', in Fig. 36, we easily discover that

$$u' = u \cos\beta + v \sin\beta,$$
$$v' = v \cos\beta - u \sin\beta;$$
$$\therefore \quad B^{u+iv} = b^{u\cos\beta + v\sin\beta} \cdot \operatorname{cis} \frac{v\cos\beta - u\sin\beta}{m}.$$

Or since $b = e^{1/m}$ (Art. 24), this formula may be written

$$B^{u+iv} = e^{(u\cos\beta + v\sin\beta)/m} \cdot \operatorname{cis} \frac{v\cos\beta - u\sin\beta}{m}.$$

When $\beta = 0$, it becomes

$$b^{u+iv} = b^u \left(\cos\frac{v}{m} + i \sin\frac{v}{m}\right),$$

and when $m = 1$ and therefore $b = e$, it assumes the more special form

$$e^{u+iv} = e^u (\cos v + i \sin v),$$

an equation due to Euler.*

* *Introductio in Analysin Infinitorum*, ed. Nov., 1797, Lib. I, p. 104.

74. Demoivre's Theorem. When $u = 0$, Euler's formula becomes

$$e^{iv} = \operatorname{cis} v,$$

and by involution

$$e^{inv} = (\operatorname{cis} v)^n = \operatorname{cis} nv,$$

or $$(\cos v + i \sin v)^n = \cos nv + i \sin nv$$

for all real values of n. This equation is known as Demoivre's theorem.*

75. Relations between Base and Modulus. Let the lines EN', EQ and OF be regarded for the moment as rigidly connected with one another and be turned conjointly in the plane about the fixed point O through an arbitrary angle, whose arc-ratio may here be denoted by γ. ON', in the new position thus given it, then forms with OJ an angle whose arc-ratio is $\beta + \gamma$, the modulus κ, $= m \operatorname{cis} \beta$, by virtue of this change, becomes

$$m \operatorname{cis} (\beta + \gamma) = m \operatorname{cis} \beta \cdot \operatorname{cis} \gamma = \kappa \cdot \operatorname{cis} \gamma,$$

and since $\overline{OQ}$, in common with the other lines with which it is connected, is turned about O through the angle of arc-ratio γ, w is hereby transformed into $w \operatorname{cis} \gamma$; while the locus of P is in no way disturbed by any of these changes. Hence

$$w \operatorname{cis} \gamma = \operatorname{cis} \gamma \cdot \log_\kappa z = \log_{\kappa \operatorname{cis} \gamma} z.$$

In a second transformation, let the motion of P still remain undisturbed, while the speed of Q is changed from μ to $n\mu$ ($n =$ a real quantity). By this change the modulus $\kappa \operatorname{cis} \gamma$ becomes $n\kappa \operatorname{cis} \gamma$, the distance of Q from the origin becomes nq instead of q, and $w \operatorname{cis} \gamma$ is transformed into $nw \operatorname{cis} \gamma$. Hence, writing $n \operatorname{cis} \gamma = \nu$, we have

* Demoivre: *Miscellanea Analytica* (Lond., 1730), p. 1.

$$vw = v \log_\kappa z = \log_{v\kappa} z,$$

in which v is any complex quantity, and we may reiterate for gonic systems of logarithms the first proposition of Art. 24:

(i). *To multiply the modulus of a logaritnm by any quantity has the effect of multiplying the logarithm itself by the same quantity.*

Corresponding to this equation connecting logarithms in two systems whose moduli are κ and $v\kappa$, the inverse, or exponential relation is

$$\exp_{v\kappa} vw = \exp_\kappa w = \exp_\kappa\left(\frac{1}{\kappa}\kappa w\right).$$

Let C be the base in the system whose modulus is $v\kappa$; then the following equations co-exist:

$$w = \log_\kappa z, \quad z = \exp_\kappa w = B^w,$$
$$vw = \log_{v\kappa} z, \quad z = \exp_{v\kappa} vw = C^{vw},$$

in which are involved, as simultaneous values of w and z,

$$w = 1, \quad \text{when } z = B,$$
$$w = 1/v, \quad \text{when } z = C. \qquad \text{(Art. 68 (ii).)}$$

These pairs of values, substituted successively in the fourth and second of the previous group of equations, give, as the relations connecting B, C and v,

$$B = C^v, \quad C = B^{1/v}.$$

Hence we may reiterate for gonic systems of logarithms the second proposition of Art. 24:

(ii). *If the modulus be changed from κ to $v\kappa$, the corresponding base is changed from B to $B^{1/v}$.*

The third proposition of Art. 24 is a corollary of this second; for if the modulus be changed from κ to 1, the base is thereby changed from B to B^{κ}; that is,

$$\exp_{\kappa} \kappa = B^{\kappa} = e; \text{ or}$$

(iii). *The exponential of any quantity with respect to itself as a modulus is equal to natural base.*

Finally, if B be substituted for z in the equation $\log_{\kappa} z = \kappa \ln z$, which is a special case of the formula of proposition (i), the resulting relations between κ and B are

$$\kappa = 1 / \ln B, \quad B = e^{1/\kappa},$$

and, in terms of its base and of natural logarithms, the logarithm to modulus κ is

$$\log_{\kappa} z = \ln z / \ln B.$$

76. The Law of Involution. By virtue of proposition (i) of last Article we have in general

$$\exp_{t\kappa} w = \exp_{\kappa} w / t,$$

and

$$\exp_{\kappa / t} w = \exp_{\kappa} tw.$$

Hence

$$\exp_{\kappa} tw = \exp_{\kappa / t} w = \exp_{\kappa / w} t = \exp_{\kappa / tw} 1.$$

Otherwise expressed, since changing κ into κ / t, or κ / w, changes B into B^{t}, or B^{w}, the statement contained in this set of equations is that

$$B^{tw} = (B^{t})^{w} = (B^{w})^{t}.$$

Obviously t or w, or both, may be replaced by their reciprocals in this formula, and the law of involution, more completely stated, is

$$(B^{\times t})^{\times w} = (B^{\times w})^{\times t}. \qquad \text{(Cf. Art. 25.)}$$

77. The Law of Metathesis. Let $z = B^w$; then by the law of involution

$$z^t = (B^w)^t = B^{wt},$$

and to these there correspond the inverse relations

$$w = \log_K z,$$
$$wt = \log_K z^t;$$

whence the law of metathesis,

$$t \log_K z = \log_K z^t. \qquad \text{(Cf. Art. 26.)}$$

Also, by changing the modulus (Art. 75) we may write

$$t \log_w z = w \log_t z. \qquad \text{(Cf. Art. 36.)}$$

78. The Law of Indices. Let w and t be any two complex quantities, $w = u + iv$, $t = r + is$, in which u, v, r and s are real. By the exponential formula (Art. 73),

$$B^w = b^{u\cos\beta + v\sin\beta} \operatorname{cis} \frac{v\cos\beta - u\sin\beta}{m},$$

$$B^t = b^{r\cos\beta + s\sin\beta} \operatorname{cis} \frac{s\cos\beta - r\sin\beta}{m},$$

in which m and b correspond to one another as modulus and base respectively in an agonic system of logarithms, and are both real. Hence, by the laws of geometric multiplication and division (Arts. 60, 61),

$$B^w \times\!/ B^t = b^{(u\pm r)\cos\beta + (v\pm s)\sin\beta} \operatorname{cis} \frac{(v\pm s)\cos\beta - (u\pm r)\sin\beta}{m}$$

$$= B^{(u\pm r) + i(v\pm s)}.$$

But, by the laws of geometric addition and subtraction (Art. 58),

$$(u \pm r) + i(v \pm s) = (u + iv) \pm (r + is)$$
$$= w \pm t;$$
$$\therefore \quad B^w \times B^t = B^{w \pm t}.$$

Thus the law of indices obtains for complex quantities. (Cf. Art. 27.)

79. **The Addition Theorem.** Operating upon both sides of the equation last written with $\log_\kappa$, we have

$$\log_\kappa (B^w \times B^t) = w \pm t,$$

which, if $B^w = z$ and $B^t = z'$, and therefore $z = \log_\kappa w$ and $z' = \log_\kappa t$, becomes

$$\log_\kappa (z \times z') = \log_\kappa z \pm \log_\kappa z',$$

which is the addition theorem for complex quantities. (Cf. Art. 28.)

80. **The Logarithmic Spiral.** The locus of P, in Fig. 36 of Art. 68, is obviously a spiral. Its polar equation may be obtained by considering the rates of change of ρ and θ, which are respectively $\lambda\rho$ and $\omega = \lambda \tan(\phi - \beta)$. By the definition of a logarithm for real quantities (Art. 23), θ is here the logarithm of ρ with respect to the modulus $\lambda \tan(\phi - \beta) / \lambda = \tan(\phi - \beta)$; or in equivalent terms,

$$\theta = \tan(\phi - \beta) \cdot \ln \rho,$$

which is the equation sought.

This locus is called the *logarithmic spiral.* It is obvious from the definition of the motion of the point that generates this curve (Art. 68), that the spiral encircles the origin an infinite number of times, coming nearer to it with every return, and that it likewise passes around the circumference of the unit circle an infinite number of times, going continually farther away from it.

81. Periodicity of Exponentials. From its character as an operator that turns any line in the plane through the angle whose arc-ratio is θ, it is evident that $\operatorname{cis}\theta$ acquires once all its possible values corresponding to real values of θ, when θ passes continuously from 0 to 2π and, k being any integer, repeats the same cycle of values through every interval from $2k\pi$ to $2(k+1)\pi$, so that for all integral values of k,

$$\operatorname{cis}(\theta + 2k\pi) = \operatorname{cis}\theta.$$

In consequence of this property, $\operatorname{cis}\theta$ is said to be *periodic*, having the *period* 2π.

The exponential B^w has similarly a period. Solving for u and v the two linear eqations,

$$\begin{aligned} u' &= u\cos\beta + v\sin\beta, \\ v' &= v\cos\beta - u\sin\beta, \end{aligned} \qquad \text{(Art. 73.)}$$

we easily find

$$\begin{aligned} u &= u'\cos\beta - v'\sin\beta \\ v &= u'\sin\beta + v'\cos\beta; \end{aligned}$$

whence

$$u + iv = (u' + iv')\operatorname{cis}\beta;$$

or this result may be inferred by direct inspection of Fig. 36. Hence the equation $B^{u+iv} = b^{u'}\operatorname{cis}v'/m$, of Art. 73, may be written

$$B^{(u'+iv')\operatorname{cis}\beta} = b^{u'}\operatorname{cis}\frac{v'}{m}.$$

In this formula let $u' = 0$ and $v' = 2km\pi$, k being any integer. Then, since $\operatorname{cis}2k\pi = 1$,

$$B^{(u'+iv')\operatorname{cis}\beta} = B^{2ki\kappa\pi} = \operatorname{cis}\frac{2km\pi}{m} = 1,$$

and therefore

$$B^{w+2ki\kappa\pi} = B^w B^{2ki\kappa\pi} = B^w.$$

Hence B^w has the period $2i\kappa\pi$. In particular e^w has the period $2i\pi$.

It is obvious that the only series of values of v' that will render $\operatorname{cis} v' \, m = 1$ is $2km\pi$, where k is an integer (Art. 63). Hence the only value of w_1 that will render $B^{w+w_1} = B^w$ is $w_1 = 2ki\kappa\pi$. The function B^w therefore has only one period and is said to be *singly* periodic.

82. Many-Valuedness of Logarithms. As a consequence of the periodicity of $B^w = z$, the logarithm (for all integral values of k) has the form

$$\log_\kappa z = w + 2ik\kappa\pi,$$

that is, $\log_\kappa z$, for a given value of z, has an indefinitely great number of values, differing from each other, in successive pairs, by $2i\kappa\pi$. The logarithm is therefore said to be many-valued; specifically its many-valuedness is infinite. We shall discover this property in other functions. In the natural agonic system,

$$\ln z = w + 2ik\pi.$$

83. Direct and Inverse Processes. In the present section and in Sec. VII we have had occasion to speak of logarithms and exponentials as inverse to one another. More explicitly, the exponential is called the *direct* function, the logarithm its *inverse*. We express the operation of inversion in general terms by letting f (or some other letter) stand for any one of the direct functional symbols used, such as B, or b, thus expressing z as a direct function of w in the form

$$z = f(w),$$

and then writing

$$w = f^{-1}(z),$$

for the purpose of expressing the fact that w is the corresponding inverse function of z. Thus when B^w takes the place of $f(w)$, $\log_\kappa z$ takes the place of $f^{-1}(z)$.

In other terms, inversion is described as that process which annuls the effect of the direct process. If $z = f(w)$, the effect of the operation f is annulled by the operation f^{-1}, thus:

$$f^{-1}(z) = f^{-1}f(w) = w.$$

In accordance with this definition the following processes are inverse to each other:

(i). Addition and subtraction:

$$(x + a) - a = x, \quad (x - a) + a = x.$$

(ii). Multiplication and division:

$$(x \times a) / a = x, \quad (x / a) \times a = x,$$

(iii). Exponentiation and logarithmic operation:

$$^{a}\log a^{x} = x, \quad a^{\,^{a}\log x} = x.$$

(iv). Involution and evolution:

$$(x^{a})^{1/a} = x, \quad (x^{1/a})^{a} = x.$$

In the use of the notation of inversion here described, its application to symbolic operation, in the form $ff^{-1}(w) = w$, must be carefully distinguished from its application to products and quotients by which from $ab = c$ is derived $a = b^{-1}c$.

Whenever the direct function is periodic its inverse is obviously many-valued; for, if p be a period of $f(w)$, so that

$$f(w + np) = f(w) = z,$$

then

$$f^{-1}(z) = w + np,$$

for all integral values of n. (Cf. Art. 82.)

84. Agenda. Reduction of Exponential and Logarithmic Forms. Prove the following:

(1). $(a+ib)^{u+iv}$
$= e^{u/2\cdot\ln(a^2+b^2)-v\tan^{-1}b/a}\operatorname{cis}[\frac{1}{2}v\ln(a^2+b^2)+u\tan^{-1}b/a]$.

(2). $(ib)^{u+iv} = e^{u\ln b-v\pi/2}\operatorname{cis}(v\ln b+\frac{1}{2}\pi u)$.

(3). $i^{u+iv} = e^{-v\pi/2}\operatorname{cis}\frac{1}{2}\pi u$.

(4). $i^i = e^{-\pi/2}$, $i^{-2i/\pi} = e$.

(5). $\log_{m+in}(x+iy) = \frac{1}{2}m\ln(x^2+y^2) - n\tan^{-1}y/x$
$+ i[\frac{1}{2}n\ln(x^2+y^2)+m\tan^{-1}y/x]$.

(6). $\log_{m+in} iy = m\ln y - \frac{1}{2}n\pi + i(n\ln y+\frac{1}{2}m\pi)$.

(7). $\log_i(-x) = i\ln x - \pi$.

(8). $\log_{-1}(-x) = -\ln x - i\pi$.

(9). $\log_i i = -\frac{1}{2}\pi$, $\log_i(-i) = \frac{1}{2}\pi$.

(10). Given $a+ib$ as the base of a system of logarithms, find the modulus and reduce it to the form $u+iv$.

(11). Express $^{a+ib}\log(x+iy)$ in the form $u+iv$ in terms of a, b, x and y.

(12). If $u' = u\cos\beta + v\sin\beta$, $v' = v\cos\beta - u\sin\beta$, $w = u+iv$ and $\kappa = m(\cos\beta+i\sin\beta)$, prove:

$$\frac{1}{2}(e^{w/\kappa}+e^{-w/\kappa}) = \cosh\frac{u'}{m}\cos\frac{v'}{m} + i\sinh\frac{u'}{m}\sin\frac{v'}{m},$$

$$\frac{1}{2}(e^{w/\kappa}-e^{-w/\kappa}) = \sinh\frac{u'}{m}\cos\frac{v'}{m} + i\cosh\frac{u'}{m}\sin\frac{v'}{m}.$$

(13). From $f(w) = aw^2+2bw+c = z$ deduce
$f^{-1}(z) = (-b\pm\sqrt{b^2-ac+az})/a$.

XIII. What Constitutes an Algebra?

85. The Cycle of Operations Complete. It is obvious from the manner in which in the preceding pages the algebraic processes as applied to complex quantities have been defined, that out of their operation in any possible combination no new forms of quantity can arise. But in particular the examples of Art. 67 illustrate this fact in an explicit way in the case of geometric addition and multiplication; and the exponential equation

$$B^{u+iv} = b^{u\cos\beta+v\sin\beta} \cdot \operatorname{cis}\frac{v\cos\beta - u\sin\beta}{m}, \qquad \text{(Art. 73.)}$$

and the logarithmic equation

$$\log_\kappa(x+iy) = \kappa \ln(x+iy) \qquad \text{(Art. 75.)}$$

$$= \kappa \ln\left(\sqrt{x^2+y^2} \cdot \frac{x+iy}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\kappa}{2}\ln(x^2+y^2) + \kappa \ln \operatorname{cis}\left(\tan^{-1}\frac{y}{x}\right),$$

or $$\log_\kappa(x+iy) = \frac{\kappa}{2}\ln(x^2+y^2) + i\kappa\tan^{-1}\frac{y}{x},$$

exhibit with equal clearness that only complex quantities can result from the application of exponential and logarithmic operations. Hence the processes of addition and subtraction, multiplication and division, involution and evolution, exponentiation and the taking of logarithms complete the cycle of operations necessary to algebra. Other operations indeed will be introduced and applied to complex quantities, such as those defined in Art. 90, but though expressed in abbreviated form as single operations, they are combinations of those already described.

86. Definition of an Algebra. When a series of elements operating upon each other in accordance with fixed laws produce only other elements belonging to the same series, they are said to constitute a *group*. Thus all positive integers, subject only to the processes of addition and multiplication, produce only positive integers, and hence form a group. Such a group is an algebra.

The effect of introducing into the arithmetic of positive integers the further processes of subtraction and division is to break the integrity of the old group and form a new one whose elements include, not only positive integers, but all rational numbers, both positive and negative, integral and fractional. A final step through evolution, or the extracting of roots, bringing with it the logarithmic operation, leads to imaginary and complex numbers—that is, numbers composed of both a real and an imaginary part.

If now, as is legitimate, we regard all reals and imaginaries as special forms of complex quantities—reals having zero imaginary parts, imaginaries having zero real parts—then, as pointed out in the preceding article, the algebraic processes of addition, subtraction, multiplication, division, evolution, involution, exponentiation, and the taking of logarithms (logarithmication), applied to complex quantities in any of their several forms, produce only other complex quantities. And hence:

The aggregate of all complex quantities—including all reals and imaginaries, both rational and irrational—operating upon each other in all possible ways by the rules of algebra, form a closed group. Such a group is again an algebra.

If, in an algebra the elements that constitute the subjects of its operations form a closed group when subjected to a complete cycle of such operations, such an algebra

may be said to be logically complete. It is incomplete if the cycle of its operations be not a closed one. An algebra, for example, that admits evolution and the logarithmic process, but precludes the imaginary and the complex quantity is logically only the fraction of an algebra. The inability of the earlier algebraists to recognize this fact made it also impossible for them to carry out the algebraic processes of evolution and the taking of logarithms to any except real and positive numbers.

With this fundamental characteristic of the algebra of complex quantities the reader will find it interesting to compare the defining principles of Peirce's linear associative algebras, outlined in his memoir on that subject,* and Cayley's observations on multiple algebra and on the definitions of algebraic operations, contained in his British Association address at Southport in 1883, and subsequently amplified in a paper published in the *Quarterly Journal of Mathematics* for 1887.**

XIV. NUMERICAL MEASURES.

87. Scale of Equal Parts. *Every magnitude, real, imaginary or complex, as represented by a straight line, can be measured by means of an arbitrary scale of equal parts, called units, and can be expressed in terms of the assumed unit by a rational number, real, imaginary or complex, with an error that is less than any assignable number.* The method may be exemplified as follows:

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* Benjamin Peirce: *Linear Associative Algebra* (Washington, 1870), or *American Journal of Mathematics*, Vol. IV (1881), p. 97.

** Cayley: *Presidential Address* in *Report of the British Association for the Advancement of Science*, for the year 1883, and on *Multiple Algebra* in the *Quarterly Journal of Mathematics* (1887), Vol. XXII, p. 270.

88. For Real Magnitudes. Suppose the scale of equal parts to be constructed, of which the arbitrary unit shall be

$$OA = AB = BC = \ldots = MN = j,$$

and suppose the outer extremity, P, of the given magnitude OP to fall between the mth and $(m+1)^{\text{th}}$ points of division

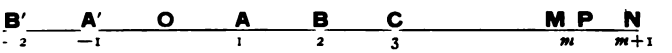

Fig. 37.

of the scale. Then m is the integral number of full unit-lengths contained in OP, and OM, or $m \times j$, differs from OP by a magnitude that is at least less than MN, that is, less than $1 \times j$, so that

$$OP = j \times m + j \times (< 1),$$

where (< 1) means 'something less than 1.'

Divide the segment MN into r equal parts and suppose P to fall between the hth and $(h+1)^{\text{th}}$ points of division. Then h is the integral number of rth parts of the unit j by which OP exceeds OM and the excess of OP over $OM + j \times (h/r)$ is less than the rth part of j; that is

$$OP = j \times \left(m + \frac{h}{r}\right) + j \times \left(< \frac{1}{r}\right).$$

Divide the rth part of MN upon which P falls into r equal parts, and suppose P to fall between the kth and $(k+1)^{\text{th}}$ points of division. Then, by the same reasoning as in the preceding case,

$$OP = j \times \left(m + \frac{h}{r} + \frac{k}{r^2}\right) + j \times \left(< \frac{1}{r^2}\right).$$

If, in the continuation of this process, P eventually falls upon one of the points of division last inserted, OP is exactly measured by a rational number of the form

$$m+\frac{h}{r}+\frac{k}{r^2}+\frac{l}{r^3}+\ldots+\frac{p}{r^q}.$$

But it may be that the process will never end. The successive rational numbers will then be more and more nearly the measures of OP, but never exactly. If the process be continued to the $(q+1)^{\text{th}}$ term, it is obvious that the error committed in assuming the rational number thus obtained as the measure of OP will be less than $1/r^q$; that is

$$OP=j\times\left(m+\frac{h}{r}+\frac{k}{r^2}+\frac{l}{r^3}+\ldots+\frac{p}{r^q}\right)+j\times\left(<\frac{1}{r^q}\right),$$

and by increasing q sufficiently this error $1/r^q$ may be made less than any number that can be assigned.

If the scale be decimal, r is 10 and this numerical measure of OP assumes the form

$$OP=j\times(m.\,hkl\ldots p)+j\times(<0.000\ldots 1),$$

a decimal number, m being the integral part of the rational term, p and 1 occupying the qth decimal place.

If the magnitude be negative, the number which is its approximate measure will be affected with the negative sign, but will differ in no other respect from the number that measures a positive magnitude of equal extent.

The common measure of the two magnitudes OP and j, here sought, may be described as their greatest common measure of the form $1/r^q$. When the process continues *ad infinitum*, they have no exact common measure.

89. For Complex Magnitudes. If the magnitude be complex it is of the form

$$OA + iAP = x + iy$$

in which x and y are real.

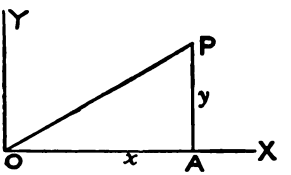

Fig. 38.

By the process just described these two real constituents may be separately measured by rational numbers in terms of an arbitrary unit, with errors that, in each case, are less than an arbitrarily small number. Thus let m and n be the rational numbers that measure x and y, with errors respectively less than ϵ and η; then

$$\begin{aligned} x + iy &= j \times (m + in) + j \times \{(< \epsilon) + i(< \eta)\} \\ &= j \times (m + in) + j \times \sqrt{\epsilon^2 + \eta^2} \left\{ \left(< \frac{\epsilon}{\sqrt{\epsilon^2 + \eta^2}} \right) + i \left(< \frac{\eta}{\sqrt{\epsilon^2 + \eta^2}} \right) \right\} \\ &= j \times (m + in) + j \times \{ < \sqrt{\epsilon^2 + \eta^2} \cdot \operatorname{cis} \phi \}, \end{aligned}$$

where $\cos \phi = \epsilon / \sqrt{\epsilon^2 + \eta^2}$, $\sin \phi = \eta / \sqrt{\epsilon^2 + \eta^2}$, and consequently $\tan \phi = \eta / \epsilon$, or $\phi = \tan^{-1}(\eta / \epsilon)$.

Since ϵ and η are separately less than an arbitrarily small real number, so is $\sqrt{\epsilon^2 + \eta^2} \cdot \operatorname{cis} \phi$ less than an arbitrarily small complex number. Hence the rational complex number $m + in$ is the measure of $x + iy$ within any required range of approximation.

CHAPTER IV.

CYCLOMETRY.

XV. CYCLIC FUNCTIONS.

90. Definitions.* The general cyclic functions, of which the circular and hyperbolic ratios of Chapter II are special forms, are defined by the following identities, in which w and w' stand for complex quantities and

$$\kappa \equiv m \cdot \operatorname{cis} \beta \equiv \text{modulus.}$$

$$(a). \quad \sin_\kappa w \equiv \frac{\kappa}{2}(e^{w/\kappa} - e^{-w/\kappa}),$$

$$(b). \quad \cos_\kappa w \equiv \frac{1}{2}(e^{w/\kappa} + e^{-w/\kappa}),$$

$$(c). \quad \tan_\kappa w \equiv \sin_\kappa w / \cos_\kappa w,$$

$$(d). \quad \cot_\kappa w \equiv \cos_\kappa w / \sin_\kappa w,$$

$$(e). \quad \sec_\kappa w \equiv 1 / \cos_\kappa w,$$

$$(f). \quad \csc_\kappa w \equiv 1 / \sin_\kappa w.$$

These are read: sine of w, cosine of w, etc., with respect to modulus κ. They may be appropriately called *modulo-cyclic*, or *modo-cyclic functions*.

When $\kappa = i$, (a) and (b) assume the special forms

$$\sin_i w = \tfrac{1}{2}(e^{iw} - e^{-iw}), \quad \cos_i w = \tfrac{1}{2}(e^{iw} + e^{-iw}),$$

* Cf. *American Journal of Mathematics*, Vol. XIV (1892), p. 192, where tentative definitions of these functions, slightly different from the above, were given.

which we now adopt as the definitions of the circular sine and cosine, and write

$$\sin w \equiv \sin_i w \equiv 1/\csc w,$$
$$\cos w \equiv \cos_i w \equiv 1/\sec w,$$
$$\tan w \equiv \tan_i w \equiv 1/\cot w.$$

When $\kappa = 1$, (a) and (b) become

$$\sin_1 w = \tfrac{1}{2}(e^{w} - e^{-w}), \quad \cos_1 w = \tfrac{1}{2}(e^{w} + e^{-w}),$$

which we adopt as the definitions of the hyperbolic sine and cosine, and write

$$\sinh w \equiv \sin_1 w \equiv 1/\operatorname{csch} w,$$
$$\cosh w \equiv \cos_1 w \equiv 1/\operatorname{sech} w,$$
$$\tanh w \equiv \tan_1 w \equiv 1/\coth w.$$

By comparing these definitions, for the arguments w and iw respectively, remembering that $i^2 = -1$ and $1/i = -i$, we find easily the following important relations:

(g).	$\sin iw = i \sinh w$,	$\sinh iw = i \sin w$,
(h).	$\cos iw = \cosh w$,	$\cosh iw = \cos w$,
(i).	$\tan iw = i \tanh w$,	$\tanh iw = i \tan w$,
(j).	$\cot iw = -i \coth w$,	$\coth iw = -i \cot w$,
(k).	$\sec iw = \operatorname{sech} w$,	$\operatorname{sech} iw = \sec w$,
(l).	$\csc iw = -i \operatorname{csch} w$,	$\operatorname{csch} iw = -i \csc w$.

91. Formulæ. From the foregoing definitions of the modocyclic functions are deduced, by the processes indicated, the following formulæ:

From (a) and (b) by addition and subtraction:

(1). $\cos_\kappa w + \kappa^{-1} \sin_\kappa w = e^{w/\kappa}$,

(2). $\cos_\kappa w - \kappa^{-1} \sin_\kappa w = e^{-w/\kappa}$.

The product of (1) and (2) gives

$$(3). \quad \cos_\kappa^2 w - \kappa^{-2} \sin_\kappa^2 w = 1.$$

By dividing (3) successively by its first and second terms:

$$(4). \quad 1 - \kappa^{-2} \tan_\kappa^2 w = \sec_\kappa^2 w.$$

$$(5). \quad \cot_\kappa^2 w - \kappa^{-2} = \csc_\kappa^2 w.$$

From (a) and (b) by carrying out the indicated multiplications, additions and subtractions:

$$(6). \quad \sin_\kappa (w \pm w') = \sin_\kappa w \cdot \cos_\kappa w' \pm \cos_\kappa w \cdot \sin_\kappa w',$$

$$(7). \quad \cos_\kappa (w \pm w') = \cos_\kappa w \cdot \cos_\kappa w' \pm \kappa^{-2} \sin_\kappa w \cdot \sin_\kappa w'.$$

From (6) and (7) by making $w' = w$:

$$(8). \quad \sin_\kappa 2w = 2 \sin_\kappa w \cdot \cos_\kappa w,$$

$$(9). \quad \cos_\kappa 2w = \cos_\kappa^2 w + \kappa^{-2} \sin_\kappa^2 w.$$

From (3) and (9) by addition and subtraction:

$$(10). \quad \cos_\kappa 2w - 1 = 2\kappa^{-2} \sin_\kappa^2 w,$$

$$(11). \quad \cos_\kappa 2w + 1 = 2 \cos_\kappa^2 w.$$

From (6) and (7) by division:

$$(12). \quad \tan_\kappa (w \pm w') = \frac{\tan_\kappa w \pm \tan_\kappa w'}{1 \pm \kappa^{-2} \tan_\kappa w \cdot \tan_\kappa w'},$$

$$(13). \quad \cot_\kappa (w \pm w') = \frac{\cot_\kappa w \pm \kappa^{-2} \tan_\kappa w'}{1 \pm \cot_\kappa w \cdot \tan_\kappa w'}.$$

From (12) and (13) by making $w' = w$:

$$(14). \quad \tan_\kappa 2w = \frac{2\tan_\kappa w}{1 + \kappa^{-1}\tan_\kappa^2 w},$$

$$(15). \quad \cot_\kappa 2w = \tfrac{1}{2}(\cot_\kappa w + \kappa^{-1}\tan_\kappa w).$$

From the two forms of (6) by addition and subtraction:

$$(16). \quad \sin_\kappa(w + w') + \sin_\kappa(w - w') = 2\sin_\kappa w \cdot \cos_\kappa w',$$

$$(17). \quad \sin_\kappa(w + w') - \sin_\kappa(w - w') = 2\cos_\kappa w \cdot \sin_\kappa w'.$$

From the two forms of (7) by addition and subtraction:

$$(18). \quad \cos_\kappa(w + w') + \cos_\kappa(w - w') = 2\cos_\kappa w \cdot \cos_\kappa w',$$

$$(19). \quad \cos_\kappa(w + w') - \cos_\kappa(w - w') = 2\kappa^{-1}\sin_\kappa w \cdot \sin_\kappa w'.$$

In (16), (17), (18), (19) write z for $w + w'$, z' for $w - w'$; they then take the respective forms:

$$(20). \quad \sin_\kappa z + \sin_\kappa z' = 2\sin_\kappa \tfrac{1}{2}(z + z') \cdot \cos_\kappa \tfrac{1}{2}(z - z'),$$

$$(21). \quad \sin_\kappa z - \sin_\kappa z' = 2\cos_\kappa \tfrac{1}{2}(z + z') \cdot \sin_\kappa \tfrac{1}{2}(z - z'),$$

$$(22). \quad \cos_\kappa z + \cos_\kappa z' = 2\cos_\kappa \tfrac{1}{2}(z + z') \cdot \cos_\kappa \tfrac{1}{2}(z - z'),$$

$$(23). \quad \cos_\kappa z - \cos_\kappa z' = 2\kappa^{-1}\sin_\kappa \tfrac{1}{2}(z + z') \cdot \sin_\kappa \tfrac{1}{2}(z - z').$$

From (12) and (13) by transposition of terms and reduction:

$$(24). \quad \tan_\kappa w \pm \tan_\kappa w' = \sin_\kappa (w \pm w') \cdot \sec_\kappa w \cdot \sec_\kappa w',$$

$$(25). \quad \cot_\kappa w \pm \cot_\kappa w' = \sin_\kappa (w' \pm w) \cdot \csc_\kappa w \cdot \csc_\kappa w'.$$

From (1) and (2) by involution:

$$(26). \quad (\cos_\kappa w' \pm \kappa^{-1} \sin_\kappa w')^w = \cos_\kappa ww' \pm \kappa^{-1} \sin_\kappa ww',$$

a generalized form of Demoivre's theorem. (Cf. Art. 74.)

All of the foregoing formulæ pass into the corresponding circular and hyperbolic special forms, by the respective substitutions $\kappa = i$ and $\kappa = 1$.

Making the proper substitutions from (g), (h), (i), (j), (k), (l) in (6) and (7), after assigning to κ the values i and 1 successively, we obtain:

$$(27). \quad \sin (w \pm iw') = \sin w \cdot \cosh w' \pm i \cos w \cdot \sinh w',$$

$$(28). \quad \cos (w \pm iw') = \cos w \cdot \cosh w' \mp i \sin w \cdot \sinh w',$$

$$(29). \quad \sinh (w \pm iw') = \sinh w \cdot \cos w' \pm i \cosh w \cdot \sin w',$$

$$(30). \quad \cosh (w \pm iw') = \cosh w \cdot \cos w' \pm i \sinh w \cdot \sin w'.$$

By definitions (a) and (b) and the definitions of circular and hyperbolic sines and cosines (Art. 90):

$$(31). \quad \sin_\kappa w = \kappa \sinh w/\kappa = i\kappa \sin w/i\kappa.$$

$$(32). \quad \cos_\kappa w = \cosh w/\kappa = \cos w/i\kappa.$$

92. Agenda. Prove the following formulæ:

(1). $\sin_\kappa \tfrac{1}{2} w = \pm\sqrt{\dfrac{\cos_\kappa w - 1}{2}}$.

(2). $\cos_\kappa \tfrac{1}{2} w = \pm\sqrt{\dfrac{\cos_\kappa w + 1}{2}}$.

(3). $\dfrac{\sin_\kappa w + \sin_\kappa w'}{\sin_\kappa w - \sin_\kappa w'} = \dfrac{\tan_\kappa \frac{1}{2}(w + w')}{\tan_\kappa \frac{1}{2}(w - w')}$.

(4). $\dfrac{\cos_\kappa w + \cos_\kappa w'}{\cos_\kappa w - \cos_\kappa w'} = \dfrac{\kappa^2 \cot_\kappa \frac{1}{2}(w + w')}{\tan_\kappa \frac{1}{2}(w - w')}$.

(5). $\dfrac{\sin_\kappa w \pm \sin_\kappa w'}{\cos_\kappa w + \cos_\kappa w'} = \tan_\kappa \tfrac{1}{2}(w \pm w')$.

(6). $\dfrac{\sin_\kappa w \pm \sin_\kappa w'}{\cos_\kappa w - \cos_\kappa w'} = \kappa^2 \cot_\kappa \tfrac{1}{2}(w \mp w')$.

(7). $\sin_\kappa 2w = \dfrac{2 \tan_\kappa w}{1 - \kappa^{-2} \tan_\kappa^2 w}$.

(8). $\sin_\kappa 3w = 3 \sin_\kappa w \cdot \cos_\kappa^2 w + \kappa^{-2} \sin_\kappa^3 w$.

(9). $\cos_\kappa 3w = 3 \kappa^{-2} \cos_\kappa w \cdot \sin_\kappa^2 w + \cos_\kappa^3 w$.

(10). $\tan_\kappa 3 w = \dfrac{3 \tan_\kappa w + \kappa^{-2} \tan_\kappa^3 w}{1 + 3 \kappa^{-2} \tan_\kappa^2 w}$.

(11). $\sec_\kappa 2w = \dfrac{\sec_\kappa^2 w}{2 - \sec_\kappa^2 w}$.

(12). $\tan_\kappa \tfrac{1}{2} w = \dfrac{\sin_\kappa w}{\cos_\kappa w + 1}$.

(13). By making $\kappa = i$ and $\kappa = 1$ successively in the formulæ of Art. 91 deduce the corresponding formulæ of the circular and hyperbolic functions.

93. Periodicity of Modocyclic Functions. In Art. 81 it was shown that all exponentials are periodic having the period $2i\kappa\pi$. The forms $e^{w/\kappa}$, e^{iw} and e^{w} have therefore the periods $2i\kappa\pi$, 2π, and $2i\pi$ respectively; hence also the modocyclic, circular and hyperbolic functions, whose definitions render them explicitly in terms of $e^{w/\kappa}$, e^{iw} and e^{w} (Art. 90) have the same respective periods, so that, if f stand for any one of the symbols $\sin_\kappa$, $\cos_\kappa$, $\tan_\kappa$, etc., g for sin, cos, etc., and h for sinh, cosh, etc., the law is, n being any integer: For modocyclic functions

$$f(w + 2ni\kappa\pi) = f(w),$$

for circular functions

$$g(w + 2n\pi) = g(w),$$

for hyperbolic functions

$$h(w + 2ni\pi) = h(w).$$

94. Agenda. Functions of Submultiples of the Periods.

(1). *The Period $i\kappa\pi$.* Show that $\tan_\kappa w$ and $\cot_\kappa w$ have also the shorter period $i\kappa\pi$.

(2). *Range of Values.* Compute the following table of the range of values of the modocyclic functions:

$w =$	$i\kappa 0$ to $\frac{i\kappa\pi}{2}$	$\frac{i\kappa\pi}{2}$ to $i\kappa\pi$	$i\kappa\pi$ to $\frac{3i\kappa\pi}{2}$	$\frac{3i\kappa\pi}{2}$ to $2i\kappa\pi$
$\sin_\kappa w$	$\mp i\kappa 0$ to $+i\kappa$	$+i\kappa$ to $\pm i\kappa 0$	$\pm i\kappa 0$ to $-i\kappa$	$-i\kappa$ to $\mp i\kappa 0$
$\cos_\kappa w$	$+1$ to ± 0	± 0 to -1	-1 to ∓ 0	∓ 0 to $+1$
$\tan_\kappa w$	$\mp i\kappa 0$ to $\pm i\kappa\infty$	$\pm i\kappa\infty$ to $\mp i\kappa 0$	$\mp i\kappa 0$ to $\pm i\kappa\infty$	$\pm i\kappa\infty$ to $\mp i\kappa 0$
$\cot_\kappa w$	$\pm i\infty/\kappa$ to $\pm i0/\kappa$	$\mp i0/\kappa$ to $\pm i\infty/\kappa$	$\pm i\infty/\kappa$ to $\mp i0/\kappa$	$\mp i0/\kappa$ to $\pm i\infty\kappa$
$\sec_\kappa w$	$+1$ to $\pm\infty$	$\pm\infty$ to -1	-1 to $\mp\infty$	$\mp\infty$ to $+1$
$\csc_\kappa w$	$\pm i\infty/\kappa$ to $-i/\kappa$	$-i/\kappa$ to $\mp i\infty/\kappa$	$\mp i\infty/\kappa$ to $+i/\kappa$	$+i/\kappa$ to $\pm i\infty/\kappa$

(3). Tabulate, as above, the range of values of the circular and hyperbolic functions.

Prove the following formulæ:

(4). $\sin_\kappa(w \pm i\kappa\pi) = -\sin_\kappa w,$

(5). $\cos_\kappa(w \pm i\kappa\pi) = -\cos_\kappa w,$

(6). $\sin_\kappa\left(\frac{i\kappa\pi}{2} \pm w\right) = i\kappa\cos_\kappa w,$

(7). $\cos_\kappa\left(\frac{i\kappa\pi}{2} \pm w\right) = \pm i/\kappa\sin_\kappa w,$

(8). $\tan_\kappa\left(\frac{i\kappa\pi}{2} \pm w\right) = \pm\kappa^2\cot_\kappa w,$

(9). $\cot_\kappa\left(\frac{i\kappa\pi}{2} \pm w\right) = \pm\kappa^{-2}\tan_\kappa w,$

(10). $\sin_\kappa\left(w \pm \frac{3i\kappa\pi}{2}\right) = \mp i\kappa\cos_\kappa w,$

(11). $\cos_\kappa\left(\frac{3i\kappa\pi}{2} \pm w\right) = \mp i/\kappa\sin_\kappa w.$

(12). Write similar formulæ for all modocyclic functions of $w \pm i\kappa\pi$, $i\kappa\pi \pm w$, $w \pm i\kappa\pi/2$, $i\kappa\pi/2 \pm w$, $w \pm 3i\kappa\pi/2$, $3i\kappa\pi/2 \pm w$, and $2i\kappa\pi \pm w$.

95. The Inverse Functions. The modocyclic, circular and hyperbolic functions, being defined in terms of exponentials, are direct; and corresponding to them, through inversion (Art. 83) we have the inverse modocyclic, circular, and hyperbolic functions.

Applying the operation of inversion to the three equations

$$z = f(w + 2ni\kappa\pi) = f(w),$$
$$z = g(w + 2n\pi) = g(w),$$
$$z = h(w + 2ni\pi) = h(w),$$

of Art. 93 we have the corresponding inverse forms:

$$f^{-1}(z) = w + 2ni\kappa\pi,$$
$$g^{-1}(z) = w + 2n\pi,$$
$$h^{-1}(z) = w + 2ni\pi,$$

in which n is any integer whatever. Thus, like logarithms, the various forms of inverse cyclic functions are many-valued to an infinite extent.

We may however define the interval over which the values of w shall range in such a way as to make it one-valued within the interval considered. Care must be taken to do this in writing the formulæ of inverse functions.

96. Agenda. Formulæ of Inverse Functions. By inverting the corresponding direct formulæ prove the following, and assign in each case the interval for which the formula is true.

(1). $\sin_\kappa^{-1} x = \cos_\kappa^{-1} \sqrt{1 + \kappa^{-2} x^2}$.

(2). $\sin_\kappa^{-1} x = \tan_\kappa^{-1} \dfrac{x}{\sqrt{1 + \kappa^{-2} x^2}}$.

(3). $\sin_\kappa^{-1} x \pm \sin_\kappa^{-1} y$
$$= \sin_\kappa^{-1} (x \sqrt{1 + \kappa^{-2} y^2} \pm y \sqrt{1 + \kappa^{-2} x^2}).$$

(4). $\sin_\kappa^{-1} x \pm \sin_\kappa^{-1} y$
$$= \cos_\kappa^{-1} \left(\sqrt{(1 + \kappa^{-2} x^2)(1 + \kappa^{-2} y^2)} \pm \kappa^{-2} xy\right).$$

(5). $\cos_\kappa^{-1} x \pm \cos_\kappa^{-1} y$
$$= \cos_\kappa^{-1}\left(xy \pm \sqrt{(x^2-1)(y^2-1)}\right).$$

(6). $\tan_\kappa^{-1} x \pm \tan_\kappa^{-1} y = \tan_\kappa^{-1} \dfrac{x \pm y}{1 \pm \kappa^{-2} xy}$.

(7). $\cot_\kappa^{-1} x \pm \cot_\kappa^{-1} y = \cot_\kappa^{-1} \dfrac{1 \pm \kappa^{-2} xy}{y \pm x}$.

(8). $2 \sin_\kappa^{-1} x = \sin_\kappa^{-1}\left(2x\sqrt{1 + \kappa^{-2}x^2}\right)$.

(9). $2 \sin_\kappa^{-1} x = \cos_\kappa^{-1}(1 + 2\kappa^{-2}x^2)$.

(10). $2 \cos_\kappa^{-1} x = \cos_\kappa^{-1}(2x^2 - 1)$.

(11). $2 \tan_\kappa^{-1} x = \tan_\kappa^{-1} \dfrac{2x}{1 + \kappa^{-2}x^2}$.

(12). $2 \tan_\kappa^{-1} x = \cos_\kappa^{-1} \dfrac{1 + \kappa^{-2}x^2}{1 - \kappa^{-2}x^2}$.

(13). $2 \tan_\kappa^{-1} x = \sin_\kappa^{-1} \dfrac{2x}{1 - \kappa^{-2}x^2}$.

(14). Deduce the corresponding formulæ for inverse circular and hyperbolic functions by assigning to κ the values $+i$ and $+1$ successively.

97. Logarithmic Forms of Inverse Cyclic Functions. The inverse cyclic functions are obviously logarithmic. We may obtain them as such in the following manner. Let u, v, x, y be four quantities, in general complex, such that

$$u = \log_\kappa v, \quad x = \cos_\kappa u, \quad y = \sin_\kappa u.$$

Then

$$e^{u/\kappa} = v = \cos_\kappa u + \kappa^{-1} \sin_\kappa u = x + \kappa^{-1} y,$$

$$u = \cos_\kappa^{-1} x = \sin_\kappa^{-1} y = \log_\kappa (x + \kappa^{-1} y)$$
$$= \kappa \ln (x + \kappa^{-1} y),$$

$$x = \pm \sqrt{1 + \kappa^2 y^2}, \quad \kappa^{-1} y = \pm \sqrt{x^2 - 1}.$$

(Art. 91, 3.)

$$\therefore \quad \sin_\kappa^{-1} y = \kappa \ln (y/\kappa \pm \sqrt{y^2/\kappa^2 + 1}),$$

and

$$\cos_\kappa^{-1} x = \kappa \ln (x \pm \sqrt{x^2 - 1}).$$

The logarithmic equivalent of $\tan_\kappa^{-1} z$ may be deduced from the definition $\tan_\kappa w \equiv \kappa (e^{w/\kappa} - e^{-w/\kappa}) / (e^{w/\kappa} + e^{-w/\kappa}) = z$, by solving this equation as a quadratic in $e^{w/\kappa}$. Thus,

$$\tan_\kappa w = \frac{\kappa (e^{2w/\kappa} - 1)}{e^{2w/\kappa} + 1} = z,$$

$$(\kappa - z) e^{2w/\kappa} = \kappa + z,$$

$$\therefore \quad e^{2w/\kappa} = \frac{\kappa + z}{\kappa - z},$$

whence

$$w = \tan_\kappa^{-1} z = \frac{\kappa}{2} \ln \frac{\kappa + z}{\kappa - z}.$$

Since $\cot_\kappa w = 1/x$, the corresponding formula for $\cot_\kappa^{-1} z$ is

$$\cot_\kappa^{-1} z = \frac{\kappa}{2} \ln \frac{\kappa z + 1}{\kappa z - 1}.$$

The forms for $\sec_\kappa^{-1} x$ and $\csc_\kappa^{-1} y$ are also obtained from those for $\cos_\kappa^{-1} x$ and $\sin_\kappa^{-1} y$ by changing x and y into $1/x$ and $1/y$ respectively. They are

$$\sec_\kappa^{-1} x = \kappa \ln (1/x \pm \sqrt{1/x^2 - 1}).$$

$$\csc_\kappa^{-1} y = \kappa \ln (1/\kappa y \pm \sqrt{1/\kappa^2 y^2 + 1}).$$

The student can easily verify these results by making the transformations independently.

The hyperbolic forms, obtained from the foregoing by putting $\kappa = 1$ are important, and are of frequent application in the integral calculus. They are given in Art. 56. As there explained, for real values of u the positive sign before the radicals must then be chosen. The forms for the inverse circular functions, got by putting $\kappa = i$, are less frequently useful.

98. Agenda. Prove the following, n being an integer:

(1). $\sin_\kappa^{-1} \frac{1}{2}\kappa(z - z^{-1}) = \cos_\kappa^{-1} \frac{1}{2}(z + z^{-1})$
$= \log_\kappa z + 2ni\kappa\pi.$

(2). $\tan_\kappa^{-1} \kappa(z^2 - 1)/(z^2 + 1) = \log_\kappa z + 2ni\kappa\pi.$

(3). $\tan_\kappa^{-1} x \pm \tan_\kappa^{-1} y = \frac{1}{2}\log_\kappa\left(\frac{\kappa + x}{\kappa - x} \times \frac{\kappa + y}{\kappa - y}\right) + 2ni\kappa\pi.$

(4). $\cos_\kappa^{-1}(x + iy) = m(\cos\beta \cosh^{-1}\rho - \sin\beta \cos^{-1}\sigma)$
$+ im(\sin\beta \cosh^{-1}\rho + \cos\beta \cos^{-1}\sigma),$

where

$$\rho = \pm\sqrt{(s + \sqrt{s^2 - x^2})}, \quad \sigma = \pm\sqrt{(s - \sqrt{s^2 - x^2})},$$
$$s = \tfrac{1}{2}(x^2 + y^2 + 1), \quad \text{and} \quad \kappa = m \operatorname{cis}\beta.$$

(5). $\sin_\kappa^{-1}(x + iy) = m(\cos\beta \cosh^{-1} \mathrm{P} - \sin\beta \sin^{-1} \Sigma)$
$+ im(\sin\beta \cosh^{-1} \mathrm{P} + \cos\beta \sin^{-1} \Sigma),$

where, as before, $\kappa = m \operatorname{cis}\beta$, and

$$\mathrm{P} = \pm\sqrt{(S + \sqrt{S^2 - X^2})}, \quad \Sigma = \pm\sqrt{(S - \sqrt{S^2 - X^2})},$$
$$S = \tfrac{1}{2}(X^2 + Y^2 + 1) = \tfrac{1}{2}(x^2 + y^2 + m^2)/m^2,$$
$$X = (-x\sin\beta + y\cos\beta)/m, \quad Y = (x\cos\beta + y\sin\beta)/m.$$

(6). Reduce $\tan_\kappa^{-1}(x + iy)$ to the form $u + iv$.

CHAPTER V.

GRAPHICAL TRANSFORMATIONS.

XVI. ORTHOMORPHOSIS UPON THE SPHERE.

99. Affix, Correspondence, Morphosis. Every complex quantity, defined geometrically by a vector drawn from the origin with proper length and direction, determines uniquely a point in its plane, namely the extremity of the vector; and conversely, to every point in the plane corresponds one and only one complex quantity. It is convenient therefore to assign, as the geometrical representative, or *affix*,* of a given complex quantity, a point in a plane, to note its different states by a series of affixes, and to represent a continuous change in it by a line, its *path*, in general not straight.

This relation between point and complex quantity is described as a *one-to-one correspondence*, and the spreading out upon the plane of the points or paths of a varying complex quantity is its *morphosis* in the plane, or its *planar morphosis*.

100. Stereographic Projection. By means of a stereographic projection we may establish a one-to-one correspondence between the plane and a sphere, so that a point upon the plane determines uniquely a point on the sphere, and *vice versa*; and the spreading out of all the

* C. Jordan: *Cours d'Analyse de l'École Polytechnique*, Vol. I, p. 106, Art. 116.

points or lines on the sphere that correspond to the affixes or paths of a complex quantity in the plane, will be its morphosis, more specifically its *orthomorphosis** upon the sphere. We accomplish the transformation in the following manner.

We place the sphere with its center at the origin of complex quantities in the plane. Regarding the plane as fixed in a horizontal position, all projecting lines are drawn from the upper extremity of the vertical diameter as a center of projection (Fig. 39 of Art. 101). Then if from this center of projection a straight line be drawn to any point in the plane, it will cut the spherical surface in one other point. The two points thus uniquely determined, one in the plane, the other on the sphere, are said to *correspond* to each other, or to be *corresponding* points. Thus a complex quantity may have an affix in the plane and a corresponding affix on the sphere.

Points in the plane inside the great circle in which it cuts the sphere correspond to points on the lower hemisphere; points in the plane outside this circle have their correspondents on the upper hemisphere.

101. Transformation Formulæ.** In order to connect algebraically the two forms of the complex quantity, in the plane and on the sphere, assume as the origin of co-ordinates in both systems the center of the sphere, let ξ and η be the horizontal, ζ the vertical co-ordinates of points on the sphere, x and y the co-ordinates of corre-

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*An orthomorphic transformation of the plane into the spherical surface. This kind of transformation is called orthomorphosis by Cayley: *Journal für die reine und angewandte Mathematik*, Bd. 107 (1891), p. 262, and *Quarterly Journal of Mathematics*, Vol. XXV (1891), p. 203. See also the first footnote to Art. 108.

**Cf. Klein: *Vorlesungen über das Ikosaeder*, p. 32.

sponding points in the plane. It will involve no loss of generality to assume the radius of the sphere to be 1. Its equation then is

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

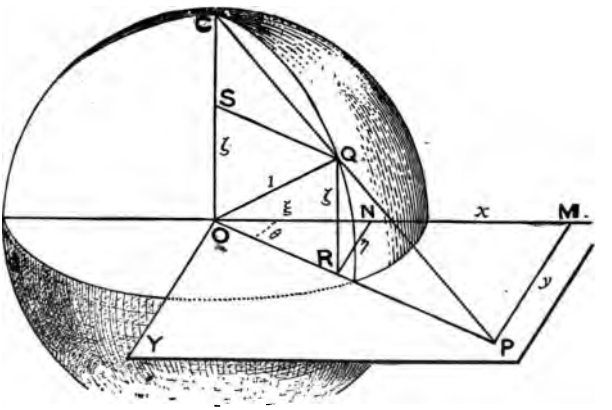

Fig. 39.

and OC being the vertical axis, P and Q corresponding points on the plane and sphere respectively, we have, by similar triangles,

$$\frac{x}{\xi} = \frac{OP}{OR} = \frac{y}{\eta} = \frac{1}{1-\zeta},$$

whence

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta},$$

and therefore

$$x + iy = \frac{\xi + i\eta}{1-\zeta}.$$

From these and the equation of the sphere we readily obtain

$$x^2+y^2+1=\frac{\xi^2+\eta^2+\zeta^2-2\zeta+1}{(1-\zeta)^2}$$
$$=\frac{2}{1-\zeta},$$

and thence the values of ζ, ξ and η in terms of x and y, namely:

$$\zeta=\frac{x^2+y^2-1}{x^2+y^2+1},$$
$$\xi=\frac{2x}{x^2+y^2+1},$$
$$\eta=\frac{2y}{x^2+y^2+1}.$$

102. The Polar Transformation. If it be desired to present the formulæ of transformation in terms of tensor and amplitude, we may write

$$x=r\cos\theta,\quad y=r\sin\theta,$$
$$\xi=\cos\phi\cdot\cos\theta,\quad \eta=\cos\phi\cdot\sin\theta,\quad \zeta=\sin\phi,$$

and accordingly

$$x+iy=r\operatorname{cis}\theta=\frac{\cos\phi}{1-\sin\phi}\cdot\operatorname{cis}\theta,$$
$$r=\frac{\cos\phi}{1-\sin\phi}.$$

Thus the expression $\cos\phi/(1-\sin\phi)\cdot\operatorname{cis}\theta$, in which ϕ and θ are independent of each other, suffices to represent all possible complex quantities.

By easy substitutions, ξ, η, ζ are found, in terms of r and θ, to be

$$\xi=\frac{2r\cos\theta}{r^2+1},\quad \eta=\frac{2r\sin\theta}{r^2+1},\quad \zeta=\frac{r^2-1}{r^2+1}.$$

103. Agenda. Properties of the Stereographic Projection.

(1). Prove analytically that a circle, or a straight line in the plane, corresponds to a circle on the sphere.

(2). Prove geometrically that any two lines in the plane cross each other at the same angle as the corresponding lines in the sphere. (Cf. Art. 107.)

(3). Show that to the centre of projection correspond all points at infinity in the plane, and that it is therefore consistent to say: there is in the complex plane but one point at infinity.

(4). Show that meridians on the sphere through the centre of projection, and parallel horizontal circles on the sphere correspond respectively to straight lines through the origin and concentric circles in the plane.

XVII. PLANAR ORTHOMORPHOSIS.

104. W-plane and Z-plane. In the graphical representation of an equation connecting two complex varying quantities w and z, it conduces to clearness of delineation and exposition to separate the figures representing the variations of w and z, and to speak of the w-plane and the z-plane as though they were distinct from one another. This language and procedure help us to see more clearly that the plane with the w-markings upon it has a distinctive character and presents in general an appearance different from that which it has when its markings represent the variations of the functions of w, and to distinguish more easily the two groups of markings from one another.

It is the purpose of the present section to describe the planar orthomorphosis of some of the functions that have been defined in the foregoing pages, that is, to cause the

point Q, the affix of w, to traverse the w-plane in a specified manner, and to mark out the paths that P, the affix of a function of w, will follow, in consequence of the assumed variations of w.

105. The Logarithmic Spirals of B^w Non-intersecting. The function B^w, is singly periodic; that is, there is only one quantity, the period $2i\kappa\pi$, multiples of which, when substituted for w, will render $B^w = 1$, (Art. 81). If now Q_0 and Q, the affixes of w_0 and w, move in the w-plane upon parallel straight lines, the variable quantities w_0 and w may be assumed to have the relation

$$w = w_0 + a,$$

where a is a constant quantity (fixed in length and direction);* and the paths of B^{w_0} and B^w will either not intersect at all, or will coincide throughout their whole extent. For, in order that the two paths may have a point in common there must be a pair of values w_0, $w_0 + a$, for which

$$B^{w_0 + a} = B^{w_0},$$

and a must be a multiple of $2i\kappa\pi$ (Art. 81). But if a be a multiple of $2i\kappa\pi$, then for all values of w_0

$$B^{w_0 + a} = B^{w_0} B^a = B^{w_0},$$

and the two w-curves have all their points common. Hence, since in the construction of Fig. 36, Art. 68, a vector representing $2i\kappa\pi$ must lie in the direction OF, we conclude:

If the paths of w_0 and w in the w-plane be parallel straight lines, the paths of B^w and B^w in the z-plane will

* This is merely a way of saying, that if a link, or rod, while remaining parallel to a fixed direction, move with one of its extremities upon a fixed straight line, its other extremity generates a second straight line parallel to the first.

be coincident, or distinct and not intersecting, according as the intercept made by the two w-lines on the modular normal is or is not a multiple of the period $2i\kappa\pi$.

106. Orthomorphosis of B^w. The fixed elements in the w-plane are the real axis, the modular line and the modular normal,—in Fig. 36, the lines OJ, ET and OF; in the z-plane they are the real axis and the unit circle.

By the operation of exponentiation, indicated by B^w, a straight line in the w-plane is transformed, metamorphosed, into a logarithmic spiral (Art. 80). Hence if the variable elements of the w-plane be assumed to be straight lines, in the z-plane they will be logarithmic spirals.

Assigning as the path of w_0 a straight line OS_0 passing through the origin (Fig. 40), write

$$w = w_0 + \alpha i\kappa,$$

in which α is a real quantity. The path of w, for a given value of α, will then be a line EC, parallel to OS_0, to which will correspond in the z-plane, a logarithmic spiral $E'C'$ (Fig 41). In particular to the path of w_0 corresponds the spiral z_0 that passes through the intersection of the real axis with the unit circle.

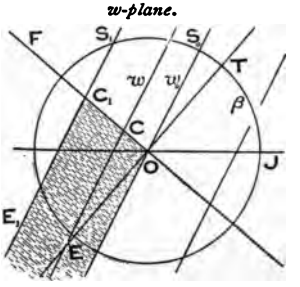

Fig. 40.

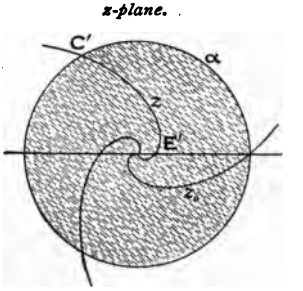

Fig. 41.

To the straight lines in the w-plane, obtained by giving different values to α, there correspond in the z-plane, so long as α is less than 2π and not less than 0, distinct non-intersecting logarithmic spirals (Art. 105). And since $OC = c = m\alpha$ (Art. 72), when α varies from 0 to 2π, c varies from 0 to $2m\pi$, or as represented in Fig. 40, from 0 to OC_1 and when EC moves from the position OS_0 to the position E_1S_1, the corresponding logarithmic spiral makes a complete revolution and sweeps over the entire z-plane.

To every point in the z-plane corresponds one and only one point in the band between the parallels OS_0, E_1S_1, whose width is $2m\pi\cos(\phi - \beta)$, and also one and only one point within every band, in the w-plane, having this width and parallel to OS_0. The construction therefore shows graphically how, to every value of z, there correspond an infinite number of values of w, namely all the values $w + 2ki\kappa\pi$, wherein k is any integer. The successive affixes of w, $w + 2i\kappa\pi$, $w - 2i\kappa\pi$, $w + 4i\kappa\pi$, etc., are obviously situated at the division-points of equidistant intervals, each equal to $2m\pi$, along a straight line through the affix of w parallel to OF.

Whenever a w-line crosses OF, the corresponding z-spiral crosses the unit circle (Art. 68). Hence, to points in the w-plane below or above the modular normal OF, correspond respectively points in the z-plane within or or without the unit circle. Thus the shaded and unshaded portions of Figs. 40, 41 correspond respectively.

The points where any spiral in the z-plane crosses the real axis are those for which B^w is real. But the necessary and sufficient condition for the vanishing of the imaginary term of B^w is

$$\sin\frac{v\cos\beta - u\sin\beta}{m} = 0, \qquad \text{(Art. 73.)}$$

for which purpose it is necessary and sufficient that

$$v\cos\beta - u\sin\beta = km\pi,$$

or

$$u\cos\left(\frac{\pi}{2}+\beta\right)+v\sin\left(\frac{\pi}{2}+\beta\right)=km\pi,$$

wherein k is any integer; and the locus of the equation last written is a straight line parallel to OT and distant from the origin a multiple of $m\pi$. Hence, whenever a w-path crosses such a line the corresponding z-spiral crosses the real axis. (Cf. Arts. 72, 80.)

As particular constructions we have: When the w-lines are parallel to the modular line OT, the z-spirals degenerate into straight lines passing through the origin; and when the w-lines are parallel to the modular normal OF, the z-spirals become circles concentric with the unit circle. (Cf. Art. 71.)

107. Isogonal Relationship. *Any two spiral paths of B^w cross one another at the same angle as the corresponding straight paths of w.*

Let θ_0, p_0, v_0, u_0 and θ, p, v', u' be two sets of corresponding values of θ, p, v' and u' (Fig. 36). It was shown in Art. 72 that $m\theta = v'$; hence

$$m(\theta-\theta_0)=v'-v_0$$

and

$$\frac{p_0(\theta-\theta_0)}{p-p_0}=\frac{p_0}{m}\cdot\frac{v'-v_0}{p-p_0}.$$

But (Fig. 36)

$$v'-v_0=(u'-u_0)\tan(\phi-\beta),$$

whence, since $p = b^{u'}$ and $p_0 = b^{u_0}$ (Art. 73),

$$\frac{p_0(\theta-\theta_0)}{p-p_0}=\frac{p_0}{m}\cdot\frac{u'-u_0}{b^{u'}-b^{u_0}}\tan(\phi-\beta);$$

and in Art. 52 it was shown that

$$\lim_{u'=u_0}\left\{\frac{b^{u'}-b^{u_0}}{u'-u_0}\right\}=\frac{p_0}{m};$$

$$\therefore \lim_{p=p_0}\left\{\frac{p_0(\theta-\theta_0)}{p-p_0}\right\}=\tan(\phi-\beta).$$

In Fig. 42 let P_0P represent a segment of the spiral of B^w, P_0T a tangent line at P_0 and let

p_0, $p \equiv OP_0$, OP,

$\theta_0 \equiv$ arc-ratio of XOP_0,

$\theta \equiv$ arc-ratio of XOP.

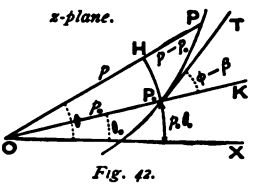

Fig. 42.

Then

$$p-p_0=HP \text{ and } p_0(\theta-\theta_0)=\overset{\frown}{P_0H};$$

whence

$$\lim_{p=p_0}\left\{\frac{p_0(\theta-\theta_0)}{p-p_0}\right\}=\tan(\text{arc-ratio of } KP_0T),$$

that is,

$$\text{arc-ratio of } KP_0T=\phi-\beta.$$

Thus the angle between the radius vector p and the spiral is constant and equal to that between the path of w and the modular line (Art. 68). It follows that the angle between any two spirals is the same as that between the two corresponding paths of w. Q. E. D.

It follows also, that to a small triangle formed by any three w-lines in the w-plane corresponds a small curvilinear triangle in the z-plane, whose angles are respectively equal to those of the former and whose sides become more and more nearly proportional to those of its correspondent as

the two triangles become smaller. The two figures therefore approach the condition of similarity to one another as a limit, when they themselves approach the vanishing point.

It is because of this property that the transformation of of w into B^w is called orthomorphic.* It is a property that can be proved to be generally true of functions of a complex variable.**

108. Orthomorphosis of $\cos_\kappa(u+iv)$. In Art. 81 it was shown that

$$u + iv = (u' + iv') \operatorname{cis} \beta,$$

and it is obvious from the definitions of Art. 90 that

$$\cos_\kappa w = \cosh \frac{w}{\kappa} = \cosh \frac{w}{m \operatorname{cis} \beta};$$

whence

$$\cos_\kappa (u + iv) = \cosh \frac{u' + iv'}{m}.$$

Here u' and v' have the significations attached to them in Art. 68, Fig. 36. For brevity write

$$u'/m = r, \quad v'/m = s.$$

By formula (30) of Art. 91 we have

$$\cosh (r + is) = \cosh r \cos s + i \sinh r \sin s.$$

Let

$$x \equiv \cosh r \cos s, \quad y \equiv \sinh r \sin s;$$

then

$$\frac{x^2}{\cosh^2 r} + \frac{y^2}{\sinh^2 r} = 1,$$

and

$$\frac{x^2}{\cos^2 s} - \frac{y^2}{\sin^2 s} = 1.$$

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* "ORTHOMORPHIC. . . . Preserving the true or original shape of the infinitesimal parts, though it may be expanding or contracting them unequally." (*Century Dictionary.*)

**Byerly: *Integral Calculus*, 2d ed., Art. 211, p. 275.

In these equations x and y are the Cartesian co-ordinates of the varying affixes of $\cos_\kappa(u+iv)$.

As equations of loci they make the following correlations apparent: When r remains constant and s varies, the affix of $u+iv$ describes a straight line parallel to the modular normal while the affix of $\cos_\kappa(u+iv)$ moves upon an ellipse; when s remains constant and r varies, the affix of $u+iv$ describes a straight line parallel to the modular line while the affix of $\cos_\kappa(u+iv)$ moves upon a hyperbola.

109. By Confocal Ellipses. The former of the last two equations represents, for $r=$ constant, an ellipse whose principal semi-diameters are $\cosh r$ and $\sinh r$. Since $\cosh^2 r - \sinh^2 r = 1$, any two or more of the ellipses obtained by giving different values to r are confocal,* having their foci on the real axis at unit's distance on either side of the origin.

If r be regarded as a parameter and be allowed to vary from 0 to $+\infty$, the ellipse represented by the equation $x^2/\cosh^2 r + y^2/\sinh^2 r = 1$, starting from the segment of straight line joining the foci as its initial form,* expands and sweeps over the entire plane. Since the equation is the same whether r be positive or negative, the variation of r from 0 to $-\infty$ gives a like result. It is only necessary, therefore, in order to make x and y pass once through all real values, to retain the positive values of r.

In any one of the ellipses the variations of x and y are governed by the variations of $\cos s$ and $\sin s$; real values of x lie between $-\cosh r$ and $+\cosh r$, real values of y between $-\sinh r$ and $+\sinh r$. To the values

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* Charles Smith: *Conic Sections*, Art. 221, p. 244.

$$s = 0,\ \frac{1}{2}\pi,\ \pi,\ \frac{3}{2}\pi,\ 2\pi,$$

correspond respectively

$$x = \cosh r,\quad 0,\quad -\cosh r,\quad 0,\quad \cosh r,$$
$$y = 0,\quad \sinh r,\quad 0,\quad -\sinh r,\quad 0.$$

By comparing these limits of variation we easily discover, r remaining constant and positive, that when s passes through each successive quadrant of a unit circle, the point (x, y) describes, in the same order, the successive quadrants of an ellipse. If r be negative, the same ellipse reappears, but its periphery is described in the reverse direction. Thus the affix of $\cosh(r + is)$ describes the entire ellipse when s passes from 0 to 2π; or obviously also, when s passes from any value s_0 to $s_0 + 2\pi$.

If r and s be now replaced by u'/m and v'/m and $\cosh(r + is)$ by $\cos_\kappa(u + iv)$, the results of this orthomorphosis may be collated as follows:

When u' varies from 0 to $+\infty$ and v' from some fixed value v_0 to $v_0 + 2m\pi$, the affix of $\cos_\kappa(u + iv)$ ranges over the entire z-plane; that is, to the band AA in the w-plane (Fig. 43), of width $2m\pi$, extending from the

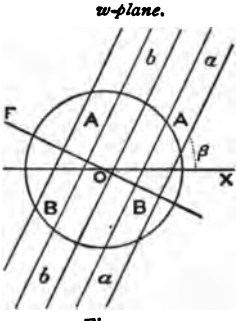

Fig. 43.

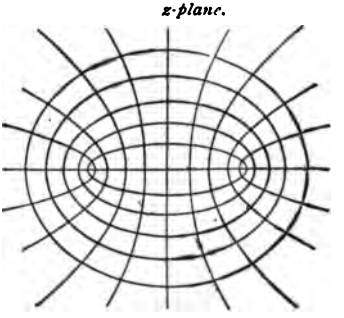

Fig. 44.

modular normal OF upwards to infinity parallel to the modular line, corresponds the entire z-plane (Fig. 44).

In like manner to the band BB, below the modular normal, and also to every other rectangular band congruent to AA having its base in the modular normal, corresponds the entire z-plane.

110. By Confocal Hyperbolas. By an analysis similar to that of Art. 100 we derive from the equation

$$\frac{x^2}{\cos^2 s} - \frac{y^2}{\sin^2 s} = 1$$

a series of confocal hyperbolas, having $\cos s$ and $\sin s$ as their principal semi-axes and, since $\cos^2 s + \sin^2 s = 1$, having their foci at the points $+1$ and -1, and forming therefore with the ellipses of Art. 109 a series of confocal conics.*

Any hyperbola of the series may be generated by giving to s the four values $2\pi - s$, s, $\pi + s$, $\pi - s$ in succession and in each case allowing r to vary from 0 to $+\infty$; the values $2\pi - s$ and s produce one branch of the hyperbola, $\pi + s$ and $\pi - s$ the other. If then, in addition to the variations of r, s pass continuously from s_0 to $s_0 + \frac{1}{2}\pi$, the series of hyperbolas will pass over the whole of the z-plane and the affix of $\cosh(r + is)$ will range over the band AA, Fig. 43. This co-ordination of the two figures is thus the same as that established in Art. 109.

But the co-ordination may take place in other ways. It is possible, for example, to generate one entire branch of

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*As is well known, these confocal conics cut one another orthogonally. (See Charles Smith's *Conic Sections*. Art. 224, p. 246.) In other words, the transformation of $u+iv$ into $\cos_K(u+iv)$ is orthomorphic; and in virtue of this orthomorphism, when the path of $u+iv$ is any straight line, that of $\cos_K(u+iv)$ is a spiral that cuts the ellipses, and also the hyperbolas, at a constant angle.

any hyperbola of the series by keeping s fixed and varying r from $-\infty$ to $+\infty$, x passing from $\cos s$ to $+\infty$, y from $-\infty$ through o to $+\infty$. The other branch of the same hyperbola is then obtained by changing s into $\pi + s$ and causing r to vary again from $-\infty$ to $+\infty$. If then, in addition to the variation of r, s be given all values between s_0 and $s_0 + \frac{1}{2}\pi$, the morphosis is completed. To the z-plane in this case are co-ordinated two parallel bands, $a\,a$ and $b\,b$ of Fig. 43, having a width equal to $\frac{1}{2}m\pi$ and extending to infinity in both directions, and separated from one another by a third band of like dimensions.

111. Agenda. Problems in Orthomorphosis. Describe the graphical transformation of $u + iv$ into each of the following functions:

(1). $\sin_\kappa (u + iv)$. (2). $\tan_\kappa (u + iv)$.

(3). $\sec_\kappa (u + iv)$. (4). $(u + iv)^n$.

(5). Prove that the graphical transformation of $u + iv$ into $\cos_\kappa (u + iv)$, or into any of the other cyclic functions is an orthomorphosis.

(6). Perform upon $e^{(u+iv)/\kappa}$, $\sin_\kappa (u + iv)$, $\cos_\kappa (u + iv)$ and $\tan_\kappa (u + iv)$, the transformation of Art. 101 and trace, upon the surface of the sphere, the path of each of these functions when the affix of $u + iv$ moves upon a straight line in the plane.

CHAPTER VI.

PROPERTIES OF POLYNOMIALS.

XVIII. ROOTS OF COMPLEX QUANTITIES.

112. Definition of an nth Root. The nth root of a given quantity is defined to be such another quantity as, when multiplied by itself $n - 1$ times (used n times as a factor), will produce the given quantity. An nth root of w is denoted by $w^{1/n}$.

Throughout the discussion concerning roots, whether of quantities or equations, n is supposed to be an integer, and unless statement be made to the contrary, a positive integer.

113. Evaluation of nth Roots. *Every complex quantity has n n^{th} roots of the form*

$$r^{1/n} \operatorname{cis} \frac{2k\pi + \theta}{n};$$

in which $r^{1/n}$ is a tensor, $(2k\pi + \theta)/n$ an amplitude, and k has one of the values $0, 1, 2, \ldots, n - 1$.

Let the complex quantity, whose roots are to be investigated, be denoted by $r \operatorname{cis} \theta$. Since r is a real positive magnitude, $r^{1/n}$ has one real positive value (Art. 23). How to find this value we do not here enquire.

By Demoivre's theorem (Art. 74), for all integral values of k,

$$\left\{ r^{1/n} \operatorname{cis} \frac{2k\pi + \theta}{n} \right\}^n = r \operatorname{cis} (2k\pi + \theta) = r \operatorname{cis} \theta,$$

and the complex quantities

$$r^{1/n}\operatorname{cis}\frac{\theta}{n},\quad r^{1/n}\operatorname{cis}\frac{2\pi+\theta}{n},\quad \ldots\ldots,\quad r^{1/n}\operatorname{cis}\frac{2(n-1)\pi+\theta}{n}$$

are all different. Hence each of them is a distinct n^{th} root of $r\operatorname{cis}\theta$, and there are n of them. Thus $r\operatorname{cis}\theta$ has n n^{th} roots, which was to be proved.

No additional values are derived from $r^{1/n}\operatorname{cis}(2k\pi+\theta)/n$ by giving to k any values other than those contained in the series 0, 1, 2, ... $n-1$, and $r^{1/n}$ has only one real positive value. Hence

A complex quantity has only n n^{th} roots.

114. Agenda. Examples in the Determination of nth Roots. Prove the following:*

(1). $\sqrt{+i}=\pm\frac{1+i}{\sqrt{2}}$, $\sqrt{-i}=\pm\frac{1-i}{\sqrt{2}}$.

(2). The cube roots of -1 are -1 and $\frac{1}{2}\pm i\frac{1}{2}\sqrt{3}$.

(3). The fourth roots of -1 are $\pm\frac{1+i}{\sqrt{2}}$ and $\pm\frac{1-i}{\sqrt{2}}$.

(4). The cube roots of $1+i$ are

$$\frac{-1+i}{2^{1/3}},\quad \frac{(\sqrt{3}+1)+i(\sqrt{3}-1)}{2^{4/3}},$$

and

$$\frac{-(\sqrt{3}-1)+i(\sqrt{3}+1)}{2^{4/3}}.$$

(5). The sixth roots of $+1$ are

$$\pm 1,\quad \tfrac{1}{2}\pm i\tfrac{1}{2}\sqrt{3}\quad\text{and}\quad -\tfrac{1}{2}\pm i\tfrac{1}{2}\sqrt{3}.$$

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*Cf. Chrystal. *Algebra*, Vol. I, pp 242, 248.

(6). Find the fifth roots of $+1$ and the sixth roots of -1.

(7). If ω be one of the complex cube roots of $+1$, then

$$(1+\omega^2)^4 = \omega, \quad 1+\omega+\omega^2 = 0,$$

and

$$(1-\omega+\omega^2)^3 = (1+\omega-\omega^2)^3 = -8.$$

Show that $1+\omega^2$ is one of the twelfth roots of $+1$.

(8). The twentieth roots of $+1$ are the successive powers, from the first to the twentieth inclusive, of

$$\tfrac{1}{4}\{1+(10+2\sqrt{5})+i(\sqrt{5}-1)\}.$$

(9). Representing the complex number of example (8) by σ, show that

$$\sigma^8 - \sigma^6 + \sigma^4 - \sigma^2 + 1 = 0, \quad \sigma^{10} + 1 = 0,$$

and that σ is therefore a tenth root of -1. Show that the even powers of σ are the tenth roots of $+1$.

XIX. Polynomials and Equations.

115. Definition of Polynomial. An algebraic expression of the form

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

in which $a_0, a_1, \ldots a_n$ are any quantities not involving z, and in which the exponents of z are all integers, is called a *rational, integral polynomial in* z. In what follows it will be sufficient to speak of it more briefly as a polynomial, the qualifying adjectives being understood. The highest exponent of z contained in it is its *degree*.

116. Roots of Equations. The investigation of Art. 113 solves the problem of finding what are called the roots of the equation

$$z^n = w,$$

in which w is a known complex quantity, and shows that such an equation, which would commonly present itself in the binominal form

$$az^n + b = 0, \qquad [b/a = -w]$$

has exactly n roots.

If additional terms containing powers of z lower than the nth be introduced into this equation, the problem of its solution becomes at once difficult, or impossible. In fact, the so-called *algebraic* solution of a general algebraic equation of a degree higher than the fourth, that is, a solution involving only radicals and having a finite number of terms, is known to be impossible.*

A discussion of the methods that may be employed in solving equations is beyond the intended scope of the present work, but the so-called fundamental theorem of algebra (Art. 120), accompanied by those propositions that are prerequisite to its demonstration, find a fitting place here.

117. The Remainder Theorem. *If a polynomial of the nth degree in z be divided by $z - \gamma$, the remainder, after n successive divisions, is the result of substituting α for z in the polynomial.*

Let $f(z)$ denote the polynomial,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

* Proved to be so by Abel: *Journal für die reine und angewandte Mathematik* (1826), Bd. I, pp. 65-84, and *Œuvres completes de N. H. Abel*, nouv. ed., Vol. I, pp 66-94.

and let the division by $z-\gamma$ be performed. It is obvious that the remainder after the first division will be of a degree lower by 1 than the dividend, that each succeeding remainder will be of a degree lower by 1 than its predecessor, and that therefore the nth remainder will not involve z. If the final quotient be denoted by Q and the final remainder by R, then

$$\frac{f(z)}{z-\gamma} = Q + \frac{R}{z-\gamma},$$

in which R does not involve z; whence, by multiplying by $z-\gamma$,

$$f(z) = Q(z-\gamma) + R.$$

This equation has the properties of an identity,* and in it z may therefore have any value whatever. Accordingly, let γ be substituted for z, and let the result of this substitution in the polynomial be denoted by $f(\gamma)$; then

$$f(\gamma) \equiv Q(\gamma-\gamma) + R,$$

in which R remains unchanged from its former value, and Q, being now a polynomial in γ, is finite. Hence

$$Q(\gamma-\gamma) = 0,$$

and $$R = f(\gamma).$$ Q. E. D.

If the remainder obtained in dividing $f(z)$ by $z-\gamma$ vanish, then $f(\gamma)=0$ and γ is said to be a *root* of the equation $f(z)=0$. Hence:

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*This may be shown by actually evolving it in specific instances. Thus, if the process here described be applied to the quadratic $a_0+a_1z+a_2z^2$, the result is

$$a_0+a_1z+a_2z^2 = (a_1+a_2\gamma+a_2z)(z-\gamma)+a_0+a_1\gamma+a_2\gamma^2,$$

and the principle of this procedure is obviously general, and independent of the degree of the polynomial. It should be observed that the identity does not depend upon the process of division; we might, in fact, produce it by the processes of addition, subtraction and rearrangement of terms. The division process is used as a convenience, not by necessity.

(i). *If $f(z)$ be exactly divisible by $z - \gamma$, γ is a root of the equation $f(z) = 0$.*

Conversely, the remainder will vanish if $f(\gamma) = 0$. Hence:

(ii). *If γ be a root of the equation $f(z) = 0$, then $f(z)$ is exactly divisible by $z - \gamma$.*

118. Argand's Theorem. *If for a given value of z the polynomial of the nth degree,*

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

have a value w_0 different from zero, its coefficients a_0, a_1, a_2, . . . a_n being given quantities, real, imaginary, or complex, there exists a second value of z, of the form $x + iy$, for which the polynomial has another value w such that

$$\text{tsr } w < \text{tsr } w_0.$$

The following demonstration of this theorem is a modification of Argand's original proof.*

Let z_0 be the given value of z and let w_0 be the resulting value of the polynomial, different from zero, so that

$$w_0 = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n.$$

Add to z_0 an arbitrary complex increment z, whose tensor and amplitude are disposable at pleasure, and let w be the resulting value of the polynomial, so that

$$w = a_0 + a_1 (z_0 + z) + a_2 (z_0 + z)^2 + \ldots + a_n (z_0 + z)^n.$$

If the several powers of the binominal $z_0 + z$, in this equation be expanded by actual multiplication, or by the

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*Argand: *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (Paris, 1806), Art. 31. The demonstration was reproduced by Cauchy, in the *Journal de l'École Royale Polytechnique* (1820), Vol. XI, pp. 411-416, in his *Analyse algébrique* (1821), ch. X, and again *Exercises d'analyse et de physique mathématique*, Vol. IV, pp. 167-170.

binominal theorem, and its terms be then arranged according to the ascending powers of z, it may be written in the form

$$w = a_0 + a_1 z_0 + a_2 z_0^2 + a_3 z_0^3 + \ldots + a_n z_0^n + b_1 z + b_2 z^2 + \ldots + b_{n-1} z^{n-1} + a_n z^n,$$

in which $b_1, b_2 \ldots b_{n-1}$ involve z_0 but not z. By hypothesis a_n is not zero, but any or all of the coefficients $b_1, b_2, \ldots b_{n-1}$ may possibly vanish. Let b_m be the first that does not vanish, b_n standing for the same thing as a_n, so that

$$w = w_0 + b_m z^m + (b_{m+1} + b_{m+2} z + \ldots + b_n z^{n-m-1}) z^{m+1},$$

and let $$b_m = a \operatorname{cis} \alpha,$$

$$b_{m+1} + b_{m+2} z + \ldots + b_n z^{n-m-1} = b \operatorname{cis} \beta,$$

and

$$z = r \operatorname{cis} \theta,$$

the quantities a, α, b, β, r and θ being real. Then

$$w = w_0 + a r^m \operatorname{cis} (\alpha + m\theta) + b r^{m+1} \operatorname{cis} \{\beta + (m+1)\theta\}.$$

Since the length of any side of a closed polygon cannot be greater than the sum of the lengths of all the other sides, or in other words, since the tensor of a sum cannot be greater that the sum of the tensors of the several terms (Art. 58),

$$\therefore \quad b \leqq \operatorname{tsr} b_{m+1} + r \operatorname{tsr} b_{m+2} + \ldots + r^{n-m-1} \operatorname{tsr} b_n.$$

Hence, by diminishing r sufficiently b may be made to differ from $\operatorname{tsr} b_{m+1}$ by an arbitrarily small quantity, and a maximum limit to the variation of r may be assigned such that b shall not exceed a fixed finite value. It follows that r may be taken so small that

$$br < a, \quad \text{or} \quad br^{m+1} < ar^m.$$

Let P_0, P' and P (Fig. 45) be the respective affixes of w_0, $w_0 + ar^m \operatorname{cis}(\alpha + m\theta)$ and w, and let θ and r, which are at our disposal, be so chosen that

$$\alpha + m\theta = \operatorname{amp} w_0 + \pi,$$

and

$$ar^m < OP_0;$$

and if this disposition of r be not sufficient to make $br^{m+1} < ar^m$, let r be still further diminished until

$$br < a$$

Then, since $\operatorname{cis}(\operatorname{amp} w_0) = \operatorname{vsr} w_0$ (Art. 60),

$$w = w_0 - ar^m \operatorname{cis}(\operatorname{amp} w_0) + br^{m+1} \operatorname{cis}\{\beta + (m+1)\theta\}$$
$$= (\operatorname{tsr} w_0 - ar^m) \operatorname{vsr} w_0 + br^{m+1} \operatorname{cis}\{\beta + (m+1)\theta\},$$

and

$$br^{m+1} < ar^m < \operatorname{tsr} w_0.$$

In accordance with these relative determinations of tensors and amplitudes the positions of P_0, P' and P in the w-plane (Fig. 45) are as follows:

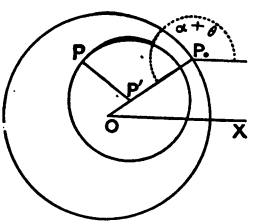

Fig. 45.

P_0, the affix of w_0, lies upon a circumference whose center is O and radius OP_0.

P', the affix of $(\operatorname{tsr} w_0 - ar^m) \operatorname{vsr} w_0$, lies upon OP_0 between O and P.

P, the affix of w, lies upon a circumference whose centre is P' and whose radius is less than $P'P_0$.

This latter circumference is therefore wholly within that upon which P_0 lies, and P is nearer to O than is P_0. But

$$OP_0 = \operatorname{tsr} w_0 \quad \text{and} \quad OP = \operatorname{tsr} w;$$

hence

$$\operatorname{tsr} w < \operatorname{tsr} w_0.$$

Q. E. D

When all the coefficients of z in the expansion of w except a_n vanish ($b_n = a_n$), the point P' is coincident with P. The final result in the foregoing demonstration remains.

119. Every Algebraic Equation has a Root. *There exists at least one value of z, real, imaginary, or complex, for which the polynomial of the nth degree,*

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

vanishes, the coefficients a_0, a_1, $\ldots$ a_n being given quantities, real, imaginary or complex.

This theorem is commonly stated in the briefer form: *Every algebraic equation has a root.* It is an immediate consequence of Argand's theorem.* For, if it be assumed that there is no value of z for which the polynomial vanishes, and if w_0 be that value of the latter whose tensor is the least possible, this hypothesis is at once contradicted by Argand's theorem which asserts that there is another value of z and a corresponding value of w such that

$$\text{tsr } w < \text{tsr } w_0.$$

Hence, tsr w must have zero as its least value, and for such a value the polynomial vanishes.

120. The Fundamental Theorem of Algebra. *A polynomial of the nth degree, such as*

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

whose coefficients a_0, a_1, $\ldots$ a_n are given real, imaginary, or complex quantities, is equal to the product of n linear

* The two propositions were not segregated by Argand; both were proved by him in the same paragraph (*loc. cit* Art. 31).

*factors multiplied by the coefficient of the highest power of z in the polynomial.**

Let $f(z)$ stand for the polynomial. By Art. 119 $f(z) = 0$ has at least one root; let that root be γ_1. Then, by Art. 117, $f(z)$ is divisible by $z - \gamma_1$ and may be expressed in the form

$$f(z) = (z - \gamma_1) f_1(z),$$

in which $f_1(z)$ is a polynomial of the degree $n - 1$. The equation $f_1(z) = 0$ has also a root; let this root be γ_2. Then, as before,

$$f_1(z) = (z - \gamma_2) f_2(z),$$

in which $f_2(z)$ is a polynomial of the degree $n - 2$; whence

$$f(z) = (z - \gamma_1)(z - \gamma_2) f_2(z).$$

This process repeated n times will produce a quotient of the degree $n - n = 0$, that is, a quantity independent of z. Hence, $f(z)$ may be expressed as the product of n linear factors and a factor independent of z; and because the coefficients of the highest powers of z on the two sides of our equation must be identical, this last factor is a_n. Thus

$$f(z) = a_n (z - \gamma_1)(z - \gamma_2) \ldots . (z - \gamma_n).$$

Q. E. D.

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* The first demonstration of this celebrated theorem was given by Gauss in his Doctor-Dissertation which bore the title *Demonstratio nova theorematis omnem functionem algebraicum rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse*, and was published at Helmstädt in 1799. He presented a second proof in December, 1815, and a third in January, 1816, to the Königliche Gesellschaft der Wissenschaften zu Göttingen. (See Gauss: *Werke*, Bd. III.) For further notices concerning this theorem see the following: H. Hankel: *Complexe Zahlen*, pp. 87-98; Burnside and Panton: *Theory of Equations*, 2d ed., pp. 442-444; Baltzer: *Elemente der Mathematik*, 6. Aufl. Bd. I, p. 299.

As a corollary of this theorem we have: *An equation of the nth degree has n and only n roots.* For, the condition necessary and sufficient in order that $f(z)$ may vanish, is that one of its linear factors shall be zero, and the putting of any one of its linear factors equal to zero gives one and only one value of z. Thus $f(z)$ will vanish for the n values $\gamma_1, \gamma_2, \ldots, \gamma_n$ of z, and for no others.

121. Agenda. Prove the following theorems concerning polynomials:

(1). Every polynomial in $x + iy$ can be reduced to the form $X + iY$.

(2). If $f(x + iy)$ be a polynomial in $x + iy$ having all its coefficients real, and if

$$f(x + iy) = X + iY,$$

then

$$f(x - iy) = X - iY.$$

(3). If all the coefficients of the polynomial $f(z)$ be real, and if

$$f(a + ib) = 0,$$

then

$$f(a - ib) = 0.$$

(4). In an algebraic equation having real coefficients imaginary roots occur in pairs.

APPENDIX.

SOME AMPLIFICATIONS.

Art. 23, page 40.

The notation b^y does not here presuppose any knowledge of the fact, easily proved as indicated in Art. 38, that b^y has the usual arithmetical meaning when b and y are numbers.

Art. 24, page 42.

The following systematic arrangement of the steps of the proof of the first proposition of page 42 will aid the student to a clearer apprehension of it.

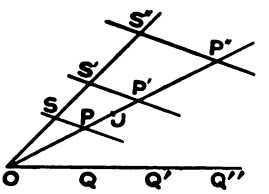
First System.	*Second System.*
base $= b$,	base $= c$,
modulus $= m$,	modulus $= km$,
$y = \log_m x$,	$ky = \log_{km} x$,
$b^y = x$,	$c^{ky} = x$,
$x = b$ when $y = 1$,	$x = c$ when $y = 1/k$.

$$\therefore \quad b^{1/k} = c \quad \text{and} \quad c^k = b.$$

Art. 27, pages 43-44.

The following alternative proof of the law of indices, though longer than that given on pages 43-44, has the merit of greater explicitness:

Let the straight line SP, while remaining transversal to the two intersecting straight lines OP, OS, move parallel to a fixed direction with a speed proportional to its perpendicular distance from O. Since OP and OS are proportional to this distance, P and S move also with speeds respectively proportional to to their distances from O.



Let Q move along a straight line through O with a constant speed μ, and let the following sets of values correspond respectively, as designated by the accents:

$$\begin{array}{llll} OP, & OP', & OP'' = x, & x', x'', \\ OS, & OS', & OS'' = s, & s', s'', \\ OQ & OQ' & OQ'' = y, & y', y''. \end{array}$$

Finally, let OS' and OJ be each equal to the linear unit and let

$$\text{speed of } P \text{ at } J = \lambda.$$

If t represent the time it takes SP to pass to the position $S''P''$, then designating ratio of distance traversed to time-interval as average speed (abbreviated to av. sp.), we have

$$\text{av. sp. of } S = (s'' - s)/t,$$
$$\text{av. sp. of } P = (x'' - x)/t,$$

and

$$(\text{av. sp. of } S)/(\text{av. sp. of } P) = (s'' - s)/(x'' - x).$$

But $$s''/s = x''/x,$$

that is, $$(s'' - s)/s = (x'' - x)/x,$$

or $$(s'' - s)/(x'' - x) = s/x,$$

$$\therefore \quad \frac{\text{av. sp. of } S}{\text{av. sp. of } P} = \frac{s}{x}.$$

This proportion remains, however small the interval be, and the limit of average speed, as the interval approaches zero, is the actual speed at the beginning of the interval. It follows that

$$\frac{\text{speed of } S}{\text{speed of } P} = \frac{s}{x} = \frac{\text{speed of } S}{\lambda x},$$

whence

$$\text{speed of } S = \lambda s,$$

and speed of S at unit's distance from O is λ. But by the definition of an exponential (Art. 23), since the positions Q', Q'' correspond respectively both to P', P'' and to S', S'',

$$x' = \exp_m y', \quad x'' = \exp_m y'',$$

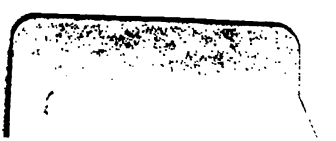
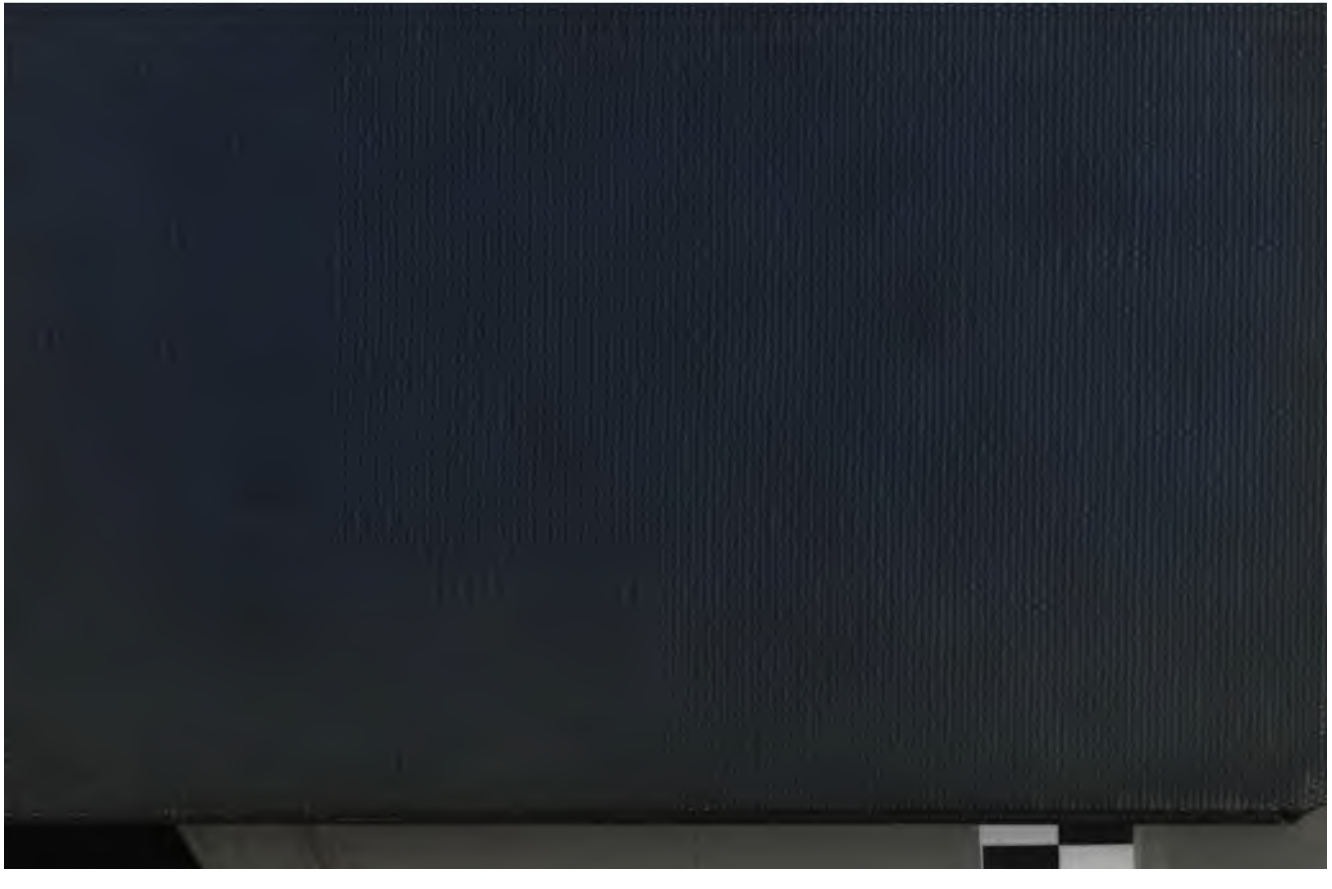
and, since $OS' = 1$,

$$s'' = \exp_m (y'' - y'),$$

where $m = \mu / \lambda$. But again, since $s' = 1$,

$$s'' = x'' / x';$$

$$\therefore \quad \exp_m y'' / \exp_m y' = \exp_m (y'' - y').$$

END OF THE UNIPLANAR ALGEBRA.

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